ELEMENTARY LINEAR ALGEBRA NOTEBOOK

Math 2270
Weber State University
Jake Watts

Math 2270, Linear Algebra (CRN–23650)

MTWF 9:30-10:20, Virtual Class, Fall 2020

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Office Hrs: Mon 11:30-12:30, Tues 9:30-10:30, Thurs 10:00-11:00 or by appointment (or drop in)
Course website: Canvas
Text: Linear Algebra and its Applications (5th Edition), by David Lay, Steven Lay, Judi McDonald

Note: This document details general course policies (such as the number of exams and weighting of various components of your course grade). For information on technical aspects of holding a virtual course, please visit the following link.

Link: Technical aspects of our online course

Homework: A homework problem list is posted on Canvas. Each week, certain sections of the homework problems will be assigned and these sections will be listed in Canvas. The assigned homework problems will be turned in on Canvas (almost always due on Friday). I will usually grade four or five of the homework problems that you submit. You are allowed one late assignment (up to one class day). After that, late homework will receive at most half credit. You must **show your work** on homework. A problem with no work shown may receive no credit.

Exams: There will be three midterm exams (in testing center) and one cumulative final exam. Tentative **dates are on Canvas and on the calendar handout**. Exams will be administered using Proctorio. This process will require students to have a computer with a working webcam.

Link: Overview of online testing using Proctorio

Link: How to check out a laptop if you do not currently have one

Make-up exams are typically not given. If, however, exceptional circumstances arise, please inform me well ahead of time of potential conflicts. In cases of university sanctioned activities or documented medical emergencies, it may be possible to arrange an alternate date for one of the midterm exams. If you miss an exam without making arrangements ahead of time, but you have an *extremely* good reason, then I may replace the corresponding test score with your percentage on the final exam.

The final exam is Mon, Dec 7 (administered via Proctorio)

University-wide policies related to fall 2020 courses:

Link: WSU webpage on student expectations for fall 2020 courses

Link: WSU webpage on course policies and procedures

Communication. As this is a virtual course, it is imperative that you frequently check your Weber State University email account and also check the course Canvas page for announcements/information regarding this course.

Calculators: On exams, you may use (and are encouraged to use) a scientific calculator, but graphing calculators are not allowed. On homework, you may use any type of calculator you want, provided that you **show your work**.

Grading[1]: Homework/Quizzes 21%, Final Exam 22%, Midterm Exams, 19% each

Grade	A- to A	B- to B+	C to C+	D- to D+
Percentage, p	$90 \le p \le 100$	$80 \le p < 90$	$70 \le p < 80$	$60 \le p < 70$

A final percentage p with $87.5 \le p < 90$ will earn a $B+$, and $80 \le p < 82.5$ will earn a $B-$. Similar statements hold for $A-$ and $C+$ grades, but a percentage of 70 or higher will earn at least a C.

Material: We will study most of chapters 1, 2, 3, 4, 5, and 6 in the text.

Attendance: You are expected to (virtually) attend every class and show up on time.

Notes: Office hours will be held virtually. Details are explained in the previously linked document on *technical aspects of our online course*. I am also very happy to answer questions by email.

Any student requiring accommodations or services due to a disability must contact Services for Students with Disabilities (SSD) in room 181 of the Student Services Center. SSD can also arrange to provide course materials (including this syllabus) in alternate formats if necessary.

Advice: I wish you all a very successful semester. Although it is likely that you and your peers have very different learning styles, my experience suggests that the following advice is usually helpful.

- In addition to time spent doing homework, spend at least 2 hours studying after every day of class.
- Be on time to every class. The first five minutes are often the most important.
- Stay awake in class. Better yet, ask questions in class.
- I encourage you to ask a questions, even if you are worried they are not good questions.
- Read the book with a pencil and paper. Work through the problems in the book along with the author. Most people cannot learn math by passively reading or watching someone do math.
- Find someone in the class with whom you can study.
- Do all of your studying long before the night before an exam. It is an ominous sign if you find yourself studying for many hours the day/night before an exam.

Contingency Plan: As this class is already a virtual class, the basic course policies and procedures will likely remain unaffected in the event of a natural disaster or global pandemic. I will continue to communicate with students via Canvas.

[1]If I feel it is warranted, I may lower the grade cutoffs slightly. For example, For example, the B- cutoff might end up being 79.4% or some other suitable number (although it is in your best interest not to expect this).

Homework Problems, Math 2270, Linear Algebra

1.1: 8, 10, 12, 14, 16, 18, 20, 24, 28

1.2: 4, 8, 10, 12, 14, 16, 20, 22, 28

1.3: 5, 8, 10, 12, 14, 18, 19, 24, 26

1.4: 6, 8, 10, 12, 13, 14, 16, 18, 22, 25

1.5: 6, 8, 10, 11, 12, 18, 20, 23, 24

1.7: 2, 6, 8, 14, 16, 18, 19, 21, 28, 32, 33, 36, 40

1.8: 2, 4, 6, 10, 12, 14, 16, 20, 21, 22

1.9: 4, 6, 7, 8, 16, 18, 20, 25, 24, 28

2.1: 2, 6, 8, 10, 16, 17, 22, 23, 28, 33 pg. 93

2.2: 4, 7a, 9, 10, 14, 16, 19, 20a, 32

2.3: 7, 8, 12, 14, 16, 17, 18, 22, 28, 34

2.5: 4, 6

3.1: 6, 8, 10, 12, 19, 20, 21, 38

3.2: 4, 6, 8, 14, 18, 20, 26, 28, 31, 36

3.3: 2, 4, 5, 6 pg. 190

4.1: 2, 6, 10, 11, 12, 14, 15, 16, 18, 23, 26

4.2: 2, 4, 10, 12, 14, 16, 19, 24, 26, 28

4.3: 4, 10, 12, 14, 16, 20, 22, 23, 31, 32

4.4: 4, 8, 10, 12, 14, 15, 16, 17, 18, 22

4.5: 4, 6, 9, 12, 14, 19, 20, 22, 26, 30

4.6: 4, 8, 10, 12, 16, 17, 18, 20, 22, 25

5.1: 2, 6, 8, 14, 15, 16, 21, 22, 31, 32

5.2: 6, 8, 12, 16, 18, 20, 21, 22

5.3: 4, 6, 14, 16, 18, 21, 22, 25, 26

6.1: 6, 10, 12, 14, 16, 18, 19, 20, 26

6.2: 2, 6, 10, 12, 14, 17, 20, 24, 27, 29

6.3: 2, 4, 8, 10, 12, 14, 16, 18, 21(a-d), 22(a-d)

6.4: 6, 9, 10

6.5: 6, 8, 12, 17(a,b), 18(a,b) This section may not be due, depending on available time.

Date (on Sunday)	Monday	Wednesday	Friday
23-Aug-19	1.1	1.2	1.3 HW 1.1, 1.2 due
30-Aug-2020	1.4	1.5	1.7 HW 1.3, 1.4 due
6-Sep-2020	Labor Day	1.7	1.8 HW 1.5, 1.7 due
13-Sep-2020	1.8, 1.9	1.9	2.1 HW 1.8, 1.9 due
20-Sep-2020	Review for Ex 1	Ex 1 Wed - Thurs Through Sec 1.9	2.2 HW 2.1 (Not due - just recomm)
27-Sep-2020	2.3	2.5 If we are behind schedule, we might skip this topic.	3.1 HW 2.2, 2.3, 2.5 due
4-Oct-2020	3.2	3.3, 4.1 (Just Cramer's Rule in 3.3)	4.1 HW 3.1, 3.2, 3.3 due
11-Oct-2020	4.2	4.2	Fall Break (no class)
18-Oct-2020	4.3	4.3	4.4 HW 4.1, 4.2 due
25-Oct-2020	Reivew for Ex 2	Ex 2 Wed - Thurs Through Sec 4.2	4.4 HW 4.3, 4.4 due
1-Nov-2020	4.5	4.6	5.1 HW 4.5, 4.6 due
8-Nov-2020	5.2	5.3	6.1 HW 5.1, 5.2, 5.3 due
15-Nov-2020	6.2	6.3	6.4 HW 6.1, 6.2 due
22-Nov-2020	6.5	Review for Ex 3	Thanksgiving holiday (no class)
29-Nov-2020	Ex 3 Mon - Tues Through Sec 6.2	Review	Review HW 6.3, 6.4 due
6-Dec-2020	Final Exam 9:30 - 11:20		

Sec.	Title	Sec.	Title
1.1	Systems of linear equations	4.1	Vector spaces and subspaces
1.2	Row reduction and echelon forms	4.2	Null spaces, column spaces, & line
1.3	Vector equations	4.3	Linearly independent sets, bases
1.4	The matrix equation Ax = b	4.4	Coordinate systems
1.5	Solution sets of linear systems	4.5	Dimension of a vector space
1.7	Linear independence	4.6	Rank
1.8	Intro to linear transformations	5.1	Eigenvectors and eigenvalues
1.9	The matrix of a linear transformatio	5.2	Characteristic equation
2.1	Matrix operations	5.3	Diagonalization
2.2	Inverse of a matrix	6.1	Inner product, length, orthogonalit
2.3	Characterizations of invertible matr	6.2	Orthogonal sets
2.5	Matrix factorizations	6.3	Orthogonal projections
3.1	Intro to determinants	6.4	Gram-Schmidt process
3.2	Properties of determinants	6.5	Least-squares problems
3.3	Cramer's rule, …		

Name: Introduction to linear Systems Textbook Section Sec 1.1

Definition: A linear equation in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form $a_1 x_1 + a_2 x_2 + \ldots a_n x_n = b$, where $a_1, a_2, \ldots a_n$ and b are real (or complex) numbers

$$\left\{\begin{array}{l} -x + y = 1 \\ 2x + y = 4 \end{array}\right\} \qquad \begin{array}{l} A = B \\ C = D \end{array} \quad \Rightarrow \quad \begin{array}{l} A + C = B + D \\ A - C = B - C \end{array}$$

$$\begin{array}{l} -x + y = 1 \\ 2x + y = 4 \end{array} \xrightarrow[2E_1 + E_2]{} \begin{array}{l} -x + y = 1 \\ 3y = 6 \end{array} \qquad \frac{3y}{3} = \frac{6}{3}, \quad y = 2$$

$$\boxed{y = 2} \qquad (x, y) = (1, 2)$$

$$-x + 2 = 1$$

$$-x = -1$$

$$\boxed{x = 1}$$

A systematic approach (that a computer could enact)

$$\left.\begin{array}{rl} x_1 \qquad - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{array}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right] \xrightarrow{\text{Replace } R_2 \text{ with } R_2 + (-2)R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

Elementary Row Operations:

Example on other Side ⇒

1. Multiply a row (equation) by a constant
2. Swap two rows
3. Replace a row with the sum of itself and a multiple of another row.

"Without mathematics, there's nothing you can do. Everything around you is mathematics. Everything around you is numbers." -Shakuntala Devi

When does a thermometer read 12 degrees more in Fahrenheit than in Celsius?

(Recall that $F = \frac{9}{5}C + 32$)

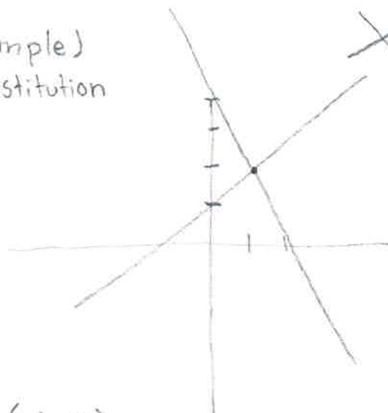
$$\begin{cases} x - \frac{9}{5}y = 32 \\ x - y = 12 \end{cases}$$

A system of linear equations

Linear equation (example) Substitution

$-x + y = 1$

$2x + y = 4$



$y = 1 + x$

$2x + 1 + x = 4$ $\quad (x,y) = (1, 2)$

$3x = 3$

$x = 1$

$y = 2$

Linear equation (example) elimination

$-x + y = 1$

$2x + y = 4$

$4(-x + y) = 4(1)$

$-4x + 4y = 4$

$4y = 4 + 4x$

$y = 1 + x$

$\frac{24}{8} - \frac{20}{8} = \frac{4}{8} = \frac{1}{2}$

Example Cont.

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right) \xrightarrow{R_2 + (-2)R_1} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{array}\right) \xrightarrow{R_3 + (-2)R_2} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{array}\right)$$

$$\xrightarrow{R_3\left(\frac{1}{5}\right)} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right) \begin{array}{l} \rightarrow x - 3z = 8 \\ \rightarrow y + 5(-1) = -2 \\ \rightarrow z = -1 \end{array}$$

$z = -1 \quad y = 3 \quad x = 5$

$(5, 3, -1)$

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & 8 & 5 & 0 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 8 & 5 & 0 \end{bmatrix}$$

$$R_3 + (-1)R_2 \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{8}R_2} \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 + (-4)R_2 \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -\frac{1}{2}x_3$

$x_2 = -\frac{5}{8}x_3$

$x_3 = x_3$

* non trivial solutions

* linearly dependent

Name: ______________________ Textbook Section 1.2

Row Reduction and Echelon Forms

A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:

1 All nonzero rows are above any rows of all zeros.

2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3. All entries in a column below a leading entry are zeros

$$\left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Row Echelon form

Reduced Row echelon form

A matrix is in reduced row echelon form if it is in row echelon form and also satisfies:

1. The leading entry in each nonzero row is 1.

2 Each leading 1 is the only nonzero entry in its column.

REF matricies (but not RREF)

$$\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \end{array}\right]$$

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A A pivot column of A that contains a pivot position

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 2 \\ 3 & 2 & 1 & -1 & 4 \end{array}\right] \sim \cdots \sim$$

* Pivot positions

Columns 1, 2, and 4 are pivot column

"The Universal Zulu Nation stands to acknowledge wisdo unity, love, and having fun, work, overcoming the negative and the wonders of God, whether we call him Allah

The "Row Reduction Algorithm"

- Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
- Select a nonzero entry in the pivot column as a pivot. If needed, interchange rows to move this entry into the pivot position.
- Use row replacement operations to create zeros in all positions below the pivot.
- Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1-3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.
- Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

$$\left.\begin{aligned} x_1 + 2x_2 + 0x_3 + 0x_4 &= 1 \\ 2x_1 + 0x_2 + x_3 + 0x_4 &= 2 \\ 3x_1 + 2x_2 + x_3 - x_4 &= 4 \end{aligned}\right\} \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 2 \\ 3 & 2 & 1 & -1 & 4 \end{array}\right]$$

$$\begin{array}{c} \xrightarrow{R_2 + -2(R_1)} \\ \xrightarrow{R_3 + -3(R_1)} \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 0 & -4 & 1 & 0 & 0 \\ 0 & -4 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_3 + (-1)R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$\begin{array}{c} \xrightarrow{-\frac{1}{4}R_2} \\ \xrightarrow{(-1)R_3} \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_1 + (-2)R_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & 0 & 1 \\ 0 & 1 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$x_1 = 1 - \frac{1}{2}x_3$

$x_2 = +\frac{1}{4}x_3$

$x_3 = \text{free}$

$x_4 = -1$

$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

$$\xrightarrow{R_2 + (-1)R_1} \left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & -2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$x_1 = -24 + 2x_3 - 3x_4$

$x_2 = -7 + 2x_3 - 2x_4$

$x_5 = 4$

x_3 = free variable

x_4 = free variable

Name: Vector Equations Textbook Section 1.3

A matrix with only one column is called a column vector, or simply a vector.

$$u = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad v = \begin{bmatrix} .2 \\ .3 \end{bmatrix} \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Given two vectors u and v in $\mathbb{R}^2$, their sum is the vector $u+v$ obtained by adding corresponding entries of u and v. For example,

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1+2 \\ -2+5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Given a vector u and a real number c, the scalar multiple of u by c is the vector cu obtained by multiplying each entry in u by c
For example,

$$\text{if } u = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \text{ and } c = 5, \text{ then } cu = 5\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$$

The number $\underline{c}$ in $\underline{c}u$ is called a scalar.

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$
(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$
(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$
(vii) $c(d\mathbf{u})=(cd)\mathbf{u}$
(viii) $1\mathbf{u}=\mathbf{u}$

The reason why we care about vectors is...

Find scalars x and y so that

$$x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ -2x \end{bmatrix} + \begin{bmatrix} 5y \\ -7y \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix} \Rightarrow \left\{ \begin{array}{l} x+5y=7 \\ -2x-7y=-5 \end{array} \right\}$$

Can we reverse this process?

$$\left\{ \begin{array}{l} 3x-5y=4 \\ x+2y=6 \end{array} \right\} \Rightarrow \begin{bmatrix} 3x-5y \\ x+2y \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \Longrightarrow \begin{bmatrix} 3x \\ x \end{bmatrix} + \begin{bmatrix} -5y \\ 2y \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$x\begin{bmatrix} 3 \\ 1 \end{bmatrix} + y\begin{bmatrix} -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

"The beauty of mathematics only shows itself to more patient followers." -Maryam Mirzakhani

$$\begin{cases} x + 0y = 3 \\ x + 2y = 2 \\ x + y = -1 \end{cases} \Rightarrow x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

1.3.5 write a system of equations that is equivalent.

$$x_1\begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}$$

$$6x_1 - 3x_2 = 1$$
$$-1x_1 + 4x_2 = -7$$
$$5x_1 + 0x_2 = -5$$

1.3.10 - write a vector equation that is equivalent to the given system of equations.

$$4x_1 + x_2 + 3x_2 = 9$$
$$x_1 - 7x_2 - 2x_2 = 2$$
$$8x_1 + 6x_2 - 5x_2 = 15$$

$$x_1\begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$$

1.3.13 - Determine if b is a linear combination of the vectors formed from the columns of the matrix A.

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix} \Longrightarrow \left[\begin{array}{ccc|c} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{array}\right] \xrightarrow{R_3+2R_1} \begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

* Violation of the Uniqueness and Existence Theorem $\Longrightarrow [0\ 0\ 0 \ldots 0\ b]$

* b is not a linear combination of the columns formed by A

1.3.14 - Determine if b is a linear combination of the vectors formed from the columns of the matrix A.

$$A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix} \xrightarrow{R_3-R_1} \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{bmatrix}$$

* Row Echelon Form

b is a linear combination of A

Name: Matrix Multiplication Textbook Section 1.4

$$\begin{bmatrix} 2 & 3 & -6 \\ 1 & -4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} =$$ linear combination of columns in the first matrix

2x3 3x1

$$(-1)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ -4 \end{bmatrix} + (-2)\begin{bmatrix} -6 \\ 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} (-1)(2)+(3)(3)+(-2)(-6) \\ (-1)(1)+(3)(-4)+(-2)(2) \end{bmatrix} = \begin{bmatrix} 19 \\ -17 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad V_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad V_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \quad V_4 = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}$$

can express $3v_1 - 8v_2 + V_3 - 7v_4$ in terms of matrix multiplication.

$$\underbrace{\begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}}_{3\times 4} \underbrace{\begin{bmatrix} 3 \\ -8 \\ 1 \\ -7 \end{bmatrix}}_{4\times 1}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

* doesn't always have a solution
* Can't always solve e.g if $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ .5 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

* does always have a solution
Span of columns of A is all of the plane $\mathbb{R}^2$

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. A has a pivot position in every row.

"Mathematics is a place where you can do things which you can't do in the real world." -Marcus du Sautoy

$$\begin{bmatrix} 3 & -8 & 5 \\ 1 & -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \pi \\ 2.7 \end{bmatrix}$$

$$x_1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -8 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} \pi \\ 2.7 \end{bmatrix}$$

$$3x_1 - 8x_2 + 5x_3 = \pi$$
$$x_1 - 2x_2 + 4x_3 = 2.7$$

1.4.6 – use the definition of Ax to write the matrix equation as a vector equation or vice versa.

$$\begin{bmatrix} 7 & -3 \\ 2 & 1 \\ 9 & -6 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$$

$$-2\begin{bmatrix} 7 \\ 2 \\ 9 \\ -3 \end{bmatrix} + -5\begin{bmatrix} -3 \\ 1 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$$

$$3x_1 + x_2 - x_3 = -6.5$$
$$-x_1 + 2x_2 + 5x_3 = 8$$
$$2x_1 + x_2 + 3x_3 = -0.5$$
$$\Downarrow$$

$$\begin{bmatrix} 3 & 1 & -1 \\ -1 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6.5 \\ 8 \\ -0.5 \end{bmatrix}$$

3x3 3x1

1.4.9 – write the system first as a vector equation and then as a matrix equation.

$$3x_1 + x_2 - 5x_3 = 9$$
$$x_2 + 4x_3 = 0$$

$$x_1\begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

vector equation

1.4.11 – Write the augmented matrix for the linear system that corresponds to the matrix equation $Ax=b$. Then solve the system and write the solution as a vector.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} -2 \\ 2 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix} \xrightarrow{R_3/5}$$

$$\begin{bmatrix} 1 & 2 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1+(-2)R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$x_1 = 0$$
$$x_2 = -3$$
$$x_3 = 1$$

$$\begin{bmatrix} 3 & 1 & -5 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$ matrix equation

$-2 + -4$

$$\begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[R_2+(-5)R_3]{R_1+(-4)R_3} \begin{bmatrix} 1 & 2 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

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Reminder

$$\left\{\begin{aligned} 2x+3y+4z&=0\\ 5x+6y+7z&=0\\ 8x+9y+10z&=0 \end{aligned}\right\}$$ is called a homogeneous System.

The zero vector $0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ is always a solution to the system $Ax=0$

We call 0 zero the trivial solution. In the above example, we have a nontrivial solution $x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column. Moreover, if the system is consistent, the solution set contains either

(i) a unique solution (when there are no free variables) or

(ii) infinitely many solutions (when there is at least one free variable).

$$\left[\begin{array}{cccc|c} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{array}\right] \quad \text{vs} \quad \left[\begin{array}{cc|c} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{array}\right] \quad \text{vs} \quad \left[\begin{array}{cccc|c} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

Infinitely many — unique Solution — Inconsistant

Typical homogeneous system:

$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & 0 \\ 3 & -7 & 8 & -5 & 8 & 0 \\ 3 & -9 & 12 & -9 & 6 & 0 \end{array}\right] \longrightarrow \cdots \longrightarrow \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\rightarrow x_1 + 0x_2 - 2x_3 + 3x_4 + 0x_5 = 0$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 - 3x_4 \\ 2x_3 - 2x_4 \\ \text{free } x_3 \\ \text{free } x_4 \\ 0 \end{bmatrix}$$

Vector form

free = x_n

IMPORTANT $\begin{bmatrix} 2x_3 - 3x_4 \\ 2x_3 - 2x_4 \\ x_3 + 0x_4 \\ 0x_3 + x_4 \\ 0x_3 + 0x_4 \end{bmatrix} \longrightarrow$ Cont.

"While physics and mathematics may tell us how the universe began, they are not much use in predicting human behavior because there are far too many equations to solve. I'm no better than anyone else at understanding what makes people tick, particularly women." -Stephen Hawking

ex. $2x_1 + 6x_2 - x_3 = 0$ $\qquad Ax = 0$

$$\begin{bmatrix} 2 & 6 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 & -1 & | & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 3 & -\frac{1}{2} & | & 0 \end{bmatrix}$$

so $x_1 = -3x_2 + \frac{1}{2}x_3$

$x_2 = x_2$

$x_3 = x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_2 + \frac{1}{2}x_3 \\ 1x_2 + 0x_3 \\ 0x_3 + 1x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

$2x_1 + 6x_2 - x_3 = 8$

$$\begin{bmatrix} 2 & 6 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 & -1 & | & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -\frac{1}{2} & | & 4 \end{bmatrix}$$

So $x_1 = -3x_2 + \frac{1}{2}x_3 + 4$

$x_2 = x_2 + 0x_3$

$x_3 = x_3 + 0x_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_2 + \frac{1}{2}x_3 + 4 \\ x_2 + 0x_3 + 0 \\ 0x_2 + x_3 + 0 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

for all real numbers x_2 and x_3

1.5.6. - write the solution set of the given homogeneous system in parametric view.

$x_1 + 3x_2 - 5x_3 = 0$

$x_1 + 4x_2 - 8x_3 = 0$

$-3x_1 - 7x_2 + 9x_3 = 0$

$$\begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix} \xrightarrow[R_3+3R_1]{R_2-R_1} \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 4x_3 = 0$

$x_2 - 3x_3 = 0$

x_3 = free

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

1.5.23.

a. A homogeneous equation is always consistent.

True - trivial solution is always a solution

b. The equation $Ax=0$ gives an explicit description of its solution set

False

c. The homogeneous equation $Ax=0$ has the trivial solution if and only if the equation has at least one free variable.

False

d. The equation $x = p + tv$ describes a line through v parallel to p

False - $x = p + tv$ describes a line p that is parallel to v

e. The solution set of $Ax=b$ is the set of all vectors of the form $w = p + v_h$, where v_h is any solution of the equation $Ax=0$

False incomplete description

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$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 - 3x_4 \\ 2x_3 - 2x_4 \\ 1x_3 + 0x_4 \\ 0x_3 + 1x_4 \\ 0x_3 + 0x_4 \end{bmatrix} = X_3 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + X_4 \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

* Solution set is just the span of $\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Answer in vector form

ex Bigger nonhomogeneous system

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 8 + (-1)x_3$

$x_2 = -4 + (-1)x_3$

$x_3 = 0 + 1x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 + (-1)x_3 \\ -4 + (-1)x_3 \\ 0 + 1x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Compare to

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -4 & -4 & -8 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Summary - For homogeneous systems $Ax = 0$:
The solution space is always a line through the origin, or a plane through the origin, or a hyperplane through the origin

1.5.10

$$\begin{bmatrix} 1 & 3 & 0 & -4 & 0 \\ 2 & 6 & 0 & -8 & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 3x_2 + 0x_3 - 4x_4 = 0$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 + 4x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$X = \begin{bmatrix} -3x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 4x_4 \\ 0 \\ 0 \\ x_4 \end{bmatrix}$$

General Solution

$$X = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Linear Independence

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$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

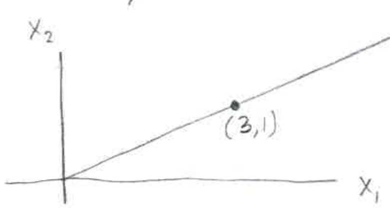
REMINDER * There are many different ways to represent the same thing.

$$\left\{ \begin{array}{l} x - y = 2 \\ 3x + y = 1 \\ x + 2y = 2 \end{array} \right\} \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 1 & 2 & 2 \end{array} \right] \Rightarrow x \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

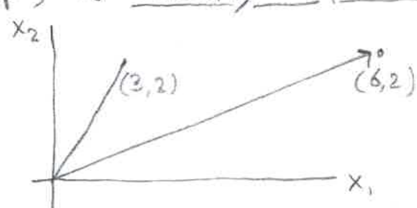
$$\begin{bmatrix} 1 & -1 \\ 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Linear independent & dependent - A set of two vectors $\{v_1, v_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

If there exist scalars $c_1 \dots c_p$, not all 0, such that $c_1v_1 + \dots + c_pv_p = 0$, then we say that the set $\{v_1, \dots v_p\}$ is linearly dependent.



Linearly dependent



Linearly independent

Theorem 8: If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $(v_1, \dots v_p)$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

ex. $\begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \end{bmatrix}$ linearly dependent ex) $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 4 & 1 & 7 & 0 \end{bmatrix} \xrightarrow{R_2+(-2)R_1} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -4 & 0 \\ 1 & 1 & 1 & 0 \\ 4 & 1 & 7 & 0 \end{bmatrix}$

$R_3+(-1)R_1 \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 4 & 1 & 7 & 0 \end{bmatrix} \xrightarrow{R_4+(-4)R_1} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$

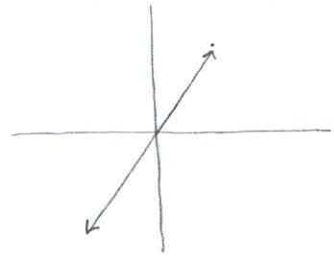
$\xrightarrow{R_4+(-1)R_3} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1R_2+R_3} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3\left(\frac{1}{3}\right)$

$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2+4R_3]{R_1+-2R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 = 0$
$x_2 = 0$
$x_3 = 0$

linearly Independent
Trivial Solution

Is the set $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -6 \end{bmatrix} \right\}$ linearly independent ?



$x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

vector equation

$\Downarrow$

System of equations $\begin{cases} x_1 - 2x_2 = 0 \\ 3x_1 - 6x_2 = 0 \end{cases}$

$\begin{bmatrix} 1 & -2 & 0 \\ 3 & -6 & 0 \end{bmatrix} R_2+(-3)R_1 \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

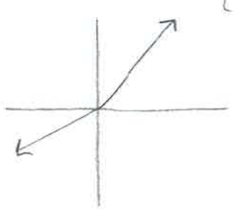
$x_1 = 2x_2$
$x_2 = x_2$ (free)

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Linearly dependent

Is the set $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \end{bmatrix} \right\}$ linearly independent?



$x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{cases} x_1 - 2x_2 = 0 \\ 3x_1 - 3x_2 = 0 \end{cases}$

$\begin{bmatrix} 1 & -2 & 0 \\ 3 & -3 & 0 \end{bmatrix} R_2+(-3)R_1 \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

$\frac{1}{3}R_2 \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1+2(R_2)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$x_1 = 0$
$x_2 = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

linearly independent

1.7.4 Determine if the vectors are linearly independent.

$\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -8 \end{bmatrix}$

$\begin{bmatrix} -1 & -2 & 0 \\ 4 & -8 & 0 \end{bmatrix} \xrightarrow{R_2+4R_1} \begin{bmatrix} 1 & -2 & 0 \\ 0 & -16 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$x_1 = 0$
$x_2 = 0$

linearly independent

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Is $\{v_1, v_2, v_3, v_4\}$ independent? $v_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 11 \\ 7 \end{bmatrix}$ $v_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ $v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$x_1 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 11 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

* Can I find a nontrivial solution?

REMEMBER

Trivial Solution means all coefficients are 0 (zero)

Non-trivial Solution means at least one coeffiencent is not 0 (zero)

$$\left[\begin{array}{cccc|c} -1 & 2 & 2 & 1 & 0 \\ 2 & 11 & 1 & 0 & 0 \\ 1 & 7 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_2 + (2)R_1} \begin{bmatrix} -1 & 2 & 2 & 1 & 0 \\ 0 & 15 & 5 & 2 & 0 \\ 1 & 7 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} -1 & 2 & 2 & 1 & 0 \\ 0 & 15 & 5 & 2 & 0 \\ 0 & 9 & 3 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{c} \frac{1}{15}R_2 \\ \frac{1}{9}R_3 \end{array} \begin{bmatrix} -1 & 2 & 2 & 1 & 0 \\ 0 & 1 & \frac{5}{15} & \frac{2}{15} & 0 \\ 0 & 1 & \frac{3}{9} & \frac{1}{9} & 0 \end{bmatrix} \xrightarrow{R_3 + (-1)R_2} \begin{bmatrix} -1 & 2 & 2 & 1 & 0 \\ 0 & 1 & \frac{5}{15} & \frac{2}{15} & 0 \\ 0 & 0 & 0 & \frac{1}{45} & 0 \end{bmatrix}$$

linearly dependent - nontrival solution

$$* \ 8v_1 - 2v_2 + 6v_3 + 0v_4 = 0$$

Theorem. (A way to identify linearly dependent sets)

Assume $v_1, \dots, v_p$ are all nonzero vectors. The set $S = \{v_1, \dots, v_p\}$ is linearly dependent if and only if some vector in S can be expressed as a linear combination of the other vectors in S.

Proof (Basic Idea) - $\{v_1, \dots v_5\}$ linearly dependent $\Longleftrightarrow$

$$\tfrac{2}{3}v_1 + 0v_2 + 5v_3 + (-\pi)v_4 + 0v_5 = 0$$

$$\pi v_4 = \tfrac{2}{3}v_1 + 5v_3$$

$$v_4 = \frac{\frac{2}{3}}{\pi} v_1 + \frac{5}{\pi} v_3$$

$$\frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc}$$

$$\frac{\frac{a}{b}}{\frac{c}{1}} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}$$

"It is generally recognized that women are better than men at languages, personal relations and multi-tasking, but less good at map-reading and spatial awareness. It is therefore not unreasonable to suppose that women might be less good at mathematics and physics." -Stephen Hawking

1.7.21 a) The columns of matrix A are linearly independent if the equation $Ax = 0$ has the trivial solution.

False

b) If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.

False

c) The columns of any 4×5 matrix are linearly dependent.

True

d) If x and y are linearly independent, and if $\{x, y, z\}$ is linearly dependent, then z is in Span $\{x, y\}$.

True

1.7.28 How many pivot columns must a 5×7 matrix have if its columns are linearly independent. Why?

5 pivot columns because there can be no free variables for the system to be independent

Theorem 4

1.7.33 - If $V_1, \dots V_4$ are in $\mathbb{R}^4$ and $V_3 = 2v_1 + V_2$, then $\{V_1, V_2, V_3, V_4\}$ is linearly dependent.

REMEMBER: Linear Dependence.

The set $\{V_1, V_p\}$ is said to be linearly dependent if there exists weights c_1, c_p such that

$$c_1 V_1 + c_2 V_2 + c_p V_p = 0$$

$$V_3 = 2v_1 + V_2 + 0v_4$$

$$0 = 2v_1 + V_2 - V_3 + 0v_4$$

Linearly dependent

1.7.32. Given $A = \begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix}$, observe that the first column plus twice the second column equals the third column. Find a nontrivial solution of $Ax = 0$.

$$x_1 + 2x_2 = x_3$$

$$x_1 + 2x_2 - x_3 = 0$$

$X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ nontrivial solution

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A set of vectors $\{v_1 \dots v_p\}$ is linearly independent if the vector equation $x_1 v_1 + \dots + x_p v_p = 0$ (where $x_1 \dots x_p$ are scalars) has only trivial solution.

If there exist scalars $c_1, \dots c_p$, not all 0, such that $c_1 v_1 + \dots + c_p v_p = 0$, then we say that the set $\{v_1, \dots, v_p\}$ is linearly dependent.

So: If you want to know if the set $\{v_1, v_2, v_3\}$ is independent, write down the vector equation $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$, and see what the solutions are.

One way to identify linearly dependent sets –

Theorem - The set $S = \{v_1, \dots v_p\}$ of nonzero vectors is linearly dependent if and only if some vector in S can be expressed as a linear combination of the other vectors in S.

* Any set of 6 or more nonzero vectors in $\mathbb{R}^5$ is linearly dependent.

Is the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ linearly independent?

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_3 + (-2)R_1} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 5 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 + (-2)R_2} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 + 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0x_1 + 0x_2 + 0x_3 = 0$ – linearly Independent

trivial solution

Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$ linearly independent?

Yes, because there are more nonzero vectors. $\mathbb{R}^3$

Theorem - For any set $\{v_1, \dots v_p\}$ of vectors in $\mathbb{R}^n$, the set must be linearly dependent if $p > n$.

"Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country." - David Hilbert

Name: ______________________________ Textbook Section 1.8

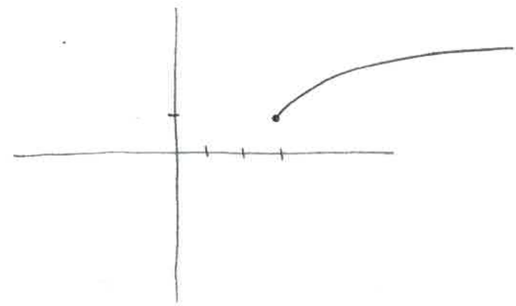
Definition -
A transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule T (a function) that assigns to each vector x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$.

We will often write $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Reminder about notation of functions

$f(x) = \sqrt{x-3} + 1$ $\qquad f: \mathbb{R}_{\geq 3} \longrightarrow \mathbb{R}$

Domain: $[3, \infty)$ $\qquad$ Range: $[1, \infty)$



Transformations

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ means that:
- T is a function
- the inputs are in $\mathbb{R}^n$, and
- the outputs are in $\mathbb{R}^m$.

Definition - A transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule T (a function) that assigns to each vector x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$.

We write $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Main example today: Take an $m \times n$ matrix A. Define

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by $T(x) = Ax$.

e.g. Suppose

$$\underbrace{T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)}_{x} = \underbrace{\begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & -2 \end{bmatrix}}_{y} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\left(\begin{bmatrix} 5 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5(1) + 1(-2) \\ 5(-1) + 1(2) \\ 5(0) + 1(-2) \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix}$$

"It's fine to work on any problem, so long as it generates interesting mathematics along the way - even if you don't solve it at the end of the day." -Andrew Wiles

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{matrix} 1(1) + (-3)(-2) = \\ 1(-1) + (-3)(2) = \\ 1(0) + (-3)(-2) = \end{matrix} \begin{bmatrix} 7 \\ -7 \\ 6 \end{bmatrix}$$

1.8.4

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ -7 \\ -9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 3 & -5 & -9 & -9 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 4 & -15 & -27 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 4 & -15 & -27 \end{bmatrix} \xrightarrow{R_3 - 4R_2} \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[R_2 + 4R_3]{R_1 - 2R_3} \begin{bmatrix} 1 & -3 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 + 3R_2}$$

$$\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$$

unique solution

Name: ______________________ Textbook Section 1.8 cont.

Definitions

- The range of a transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is the set of all vectors in $\mathbb{R}^m$ that can be written in the form $T(x)$ for some x in $\mathbb{R}^n$.
- A matrix transformation is a transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ that has the form $T(x) = Ax$ where A is some $n \times m$ matrix.

Example:

$T(x) = Ax \qquad A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 2 \end{bmatrix} \qquad T: \mathbb{R}^2 \to \mathbb{R}^3$

- compute $T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3(1) + 1(2) \\ 3(-1) + 1(1) \\ 3(2) + 1(2) \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$$

- Is $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$ in the range of T?

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

* ~~not possible~~ not even in $\mathbb{R}^3$

* not in the range

- Is $\begin{bmatrix} -1 \\ -5 \\ 2 \end{bmatrix}$ in the range of T? $\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 2 \end{bmatrix}$

* translate into augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ -1 & 1 & -5 \\ 2 & 2 & 2 \end{array}\right] \xrightarrow[R_3 + (-2)R_1]{R_2 + R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -6 \\ 0 & -2 & 4 \end{bmatrix} \xrightarrow[\frac{1}{2}R_3]{\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Yes, $\begin{bmatrix} -1 \\ -5 \\ 2 \end{bmatrix}$ is in the range of T.

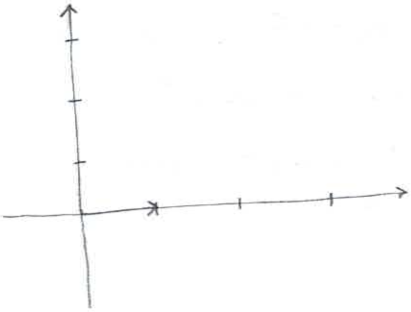
Linear transformations

Reminder

A transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$.
A matrix transformation (from $\mathbb{R}^n$ to $\mathbb{R}^m$) is a special kind of transformation (built using a matrix).

Ex.

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$ Comes from $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$



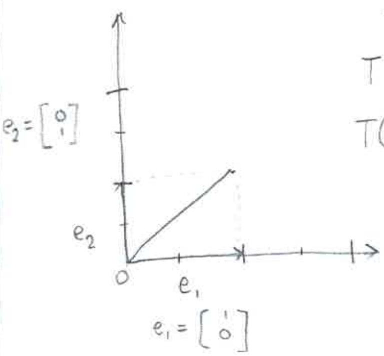
$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

Ex. 3 Lecture

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$ Comes from $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$T: \mathbb{R}^2 \to \mathbb{R}^2$



$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e_2$

$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e_1$

Name: ______________________ Textbook Section 1.8 cont.

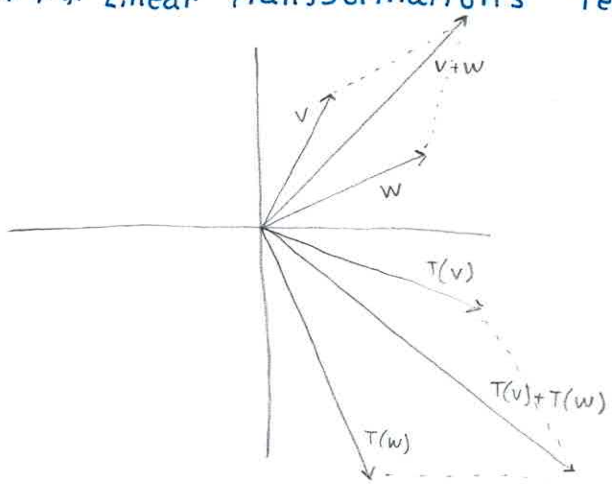
Linear transformations

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if:

1) $T(u+v) = T(u) + T(v)$ for all u and v in $\mathbb{R}^n$, and

2) $T(cv) = cT(v)$ for all c in $\mathbb{R}$ and all v in $\mathbb{R}^n$.

*** Linear transformations "respect" vector addition and scalar multiplication



* Translation is not a linear transformation

Name: ______________________________ Textbook Section 1.9

Helpful notation:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Matrix transformation:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad T(x) = Ax \qquad \underset{x}{T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)} = \underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}} \underset{x}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{Ax}{\begin{bmatrix} x + 2y \\ 3x + 4y \\ 5x + 6y \end{bmatrix}}$$

Example: $T(x) = Ax$ What is A?

what if $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - 5x_2 \\ x_1 + 3x_2 \\ -6x_1 + x_2 \end{bmatrix} \Longrightarrow T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\underset{A}{(3 \times 2)}\ \underset{x}{(2 \times 1)} = \underset{\text{output}}{3 \times 1}$$

$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix}$ $\qquad T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$

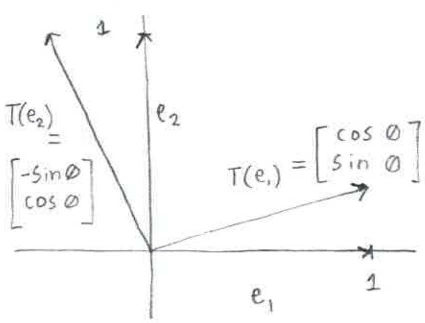
$x_1 = 1$

$x_2 = 0$

$$A = \begin{bmatrix} 3 & -5 \\ 1 & 3 \\ -6 & 1 \end{bmatrix} \qquad x_1 \begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix} = \underset{A}{\begin{bmatrix} 3 & -5 \\ 1 & 3 \\ -6 & 1 \end{bmatrix}} \underset{(x)}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

Example 2:

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ rotation of θ counter clockwise about origin



Matrix A $\quad T(x) = Ax$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotation Matrix

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One-to-one and onto

Definition: We say that $T: \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the image of at most one x in $\mathbb{R}^n$

(Never find $T(x) = T(v)$ for $x \neq v$)

One-to-one Linear Transformations

a) consider the LT with standard matrix representation

$$A = \begin{bmatrix} 1 & -3 & 4 \\ 3 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

Is this Linear transformation One-to-one?

$\Rightarrow$ Linearly independent columns? Yes $\longrightarrow$ one-to-one

No $\longrightarrow$ not one-to-one

$$\Rightarrow \begin{bmatrix} 1 & -3 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow$$

Augmented matrix

Only the trivial solution which means it's columns are Independent.

$\longrightarrow$ Yes it is one-to-one

Onto Linear Transformations

a) consider the LT with standard matrix representation

$$A = \begin{bmatrix} 1 & -3 & 4 \\ 3 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

3x3 matrix $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

Is this Linear transformation Onto?

This Linear transformation is onto if its columns span $\mathbb{R}^3$

$\Rightarrow$ To span $\mathbb{R}^3$, need 3 linearly independent vectors.

$$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

are independent (see previous example) $\Longrightarrow$ LT <u>is</u> onto

b) $A = \begin{bmatrix} 1 & -3 & 4 \\ 3 & 0 & 3 \\ 2 & 1 & 1 \end{bmatrix}$ Is this One-to-one ?

$\Rightarrow \begin{bmatrix} 1 & -3 & 4 & 0 \\ 3 & 0 & 3 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix} \xRightarrow{\text{r.r.}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Infinite # of solutions (not just trivial)

$\rightarrow$ Linearly dependent columns because some variables depend upon others

$\Rightarrow$ Not one-to-one

b) $A = \begin{bmatrix} 1 & -3 & 4 \\ 3 & 0 & 3 \\ 2 & 1 & 1 \end{bmatrix}$ $3 \times 3 \Rightarrow T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

Is this linear transformation onto ?

$\Rightarrow$ To span $\mathbb{R}^3$, need 3 linearly independent vectors.

$\Rightarrow$ Linearly dependent - review last example

This linear transformation <u>is not</u> onto

Name: ______________________ Textbook Section ~~1.9~~ 2.1

Common notation for matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad A = [a_{ij}]$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Identity matrices

<u>Matrix addition</u> Let

$$A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$$

* but $A + C$ is not defined because A and C are diffient sizes

$$2B = 2\begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}$$

$$A - 2B = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 \\ -7 & -7 & -12 \end{bmatrix}$$

$$\underset{3 \times 2}{\begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix}} \underset{2 \times 4}{\begin{bmatrix} & & \\ & C & \\ & & \end{bmatrix}} = \underset{3 \times 4}{\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}}$$

THEOREM 1 Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$
b. $(A + B) + C = A + (B + C)$
c. $A + 0 = A$
d. $r(A + B) = rA + rB$
e. $(r + s)A = rA + sA$
f. $r(sA) = (rs)A$

"The advancement and perfection of Mathematics are intimately connected with the prosperity of the State" - Napoleon

ex.

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 16 & -3 \\ 24 & 2 \end{bmatrix}$$

Name: ______________________ Textbook Section 2.1. cont

Matrix Multiplication

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \cdot \begin{bmatrix} g & h & i & j \\ k & l & m & n \\ o & p & q & r \end{bmatrix} = \underbrace{\begin{bmatrix} (ag+bk+co) & (ah+bl+cp) & (ai+bm+cq) & (aj+bn+cr) \\ (dg+ek+fo) & (dh+el+fp) & (di+em+fq) & (dj+en+fr) \end{bmatrix}}_{2\times 4 \text{ matrix}}$$

(2×3) must agree (3×4)

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

appropriate size

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left distributive law)

c. $(B + C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$ for any scalar r

e. $I_m A = A = A I_n$ (identity for matrix multiplication)

The Transpose of a Matrix

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} a & c & e & g \\ b & d & f & h \end{bmatrix}$$

4×2 $\qquad$ 2×4

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A + B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

"A Mathematician who is not also something of a poet will never be a complete mathematician" -Karl Weierstrass

Want to Solve

$$\underbrace{\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} \underset{=}{=} \underbrace{\begin{bmatrix} -6 \\ 4 \end{bmatrix}}_{b}$$

$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ 4 \end{bmatrix}$$

Name: ______________________________ Textbook Section 2.2

The Inverse of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad A = [a_{ij}]$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

$$\begin{bmatrix} 2 & 4 \\ 5 & 11 \end{bmatrix} \cdot \begin{bmatrix} \frac{11}{2} & -2 \\ -5/2 & 1 \end{bmatrix} = I_2 = \begin{bmatrix} 11-10 & -4+4 \\ \frac{55}{2} + \frac{-55}{2} & -10+11 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition – Let A be an $n \times n$ matrix. We say that a matrix B is an inverse of A if $AB = I_n$ and $BA = I_n$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

" $ad - bc$ " is called the determinant.

Theorem: An $n \times n$ matrix A is invertible if and only if A can be row reduced to the identity matrix I_n.

Proof. Basic idea ... If A is invertible, then the equation $Ax = b$ can be solved for all b in $\mathbb{R}^n$. We've seen earlier that this means A has a pivot position in every row

"The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful" Georg Cantor

$$\begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&5&0\\ 0&0&0&1 \end{bmatrix} \cdot \begin{bmatrix} a&b&c&d\\ e&f&g&h\\ i&j&k&l\\ m&n&o&p \end{bmatrix} = \begin{bmatrix} a&b&c&d\\ e&f&g&h\\ 5i&5j&5k&5l\\ m&n&o&p \end{bmatrix}$$

$$\begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&-6&0&1 \end{bmatrix} \cdot \begin{bmatrix} a&b&c&d\\ e&f&g&h\\ i&j&k&l\\ m&n&o&p \end{bmatrix} = \begin{bmatrix} a&b&c&d\\ e&f&g&h\\ i&j&k&l\\ (-6e+m)&(-6f+n)&(-6g+o)&(-6h+p) \end{bmatrix}$$

9a. In order for a matrix B to be the inverse of A, both equations $AB = I$ and $BA = I$ must be true.

True - definition of invertibility

2.2.14 - Suppose $(B-C)D = 0$, where B and C are $m \times n$ matrices and D is invertible. Show that $B = C$

$$(B-C)DD^{-1} = 0D^{-1} \qquad DD^{-1} = I$$

$$(B-C)\, I_m = 0$$

$$B - C = 0$$

$$B = C$$

2.2.16 - Suppose A and B are $n \times n$, B is invertible, and AB is invertible. Show that A is invertible.

Hint - Let $C = AB$ solve for A

$$AB = C$$

$$ABB^{-1} = CB^{-1}$$

$$AI = CB^{-1}$$

$$A = CB^{-1}$$

2.2.19. If A, B, and C are $n \times n$ invertible matrices, does the equation $C^{-1}(A+X)B^{-1} = I_n$ have a solution X? If so, find it.

$$C^{-1}(A+X)B^{-1} = I$$

$$CC^{-1}(A+X)B^{-1} = CI$$

$$I(A+X)B^{-1} = C$$

$$(A+X)B^{-1}B = CB$$

$$A + X = CB$$

$$X = CB - A$$

Name: ______________________________________ Textbook Section __________________

How to find the inverse of a square matrix of any size

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1/2 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1/2 & 0 \\ 0 & 1 & -1/2 & 1 \end{array}\right] \xrightarrow{\text{cont.}}$$

$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 3/2 & -2 \\ 0 & 1 & -1/2 & 1 \end{array}\right] \qquad A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 3/2 & -2 \\ -1/2 & 1 \end{bmatrix}$$

"The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful" Georg Cantor

Characterizations of invertible Matrices

Name: ______________________________ Textbook Section 2.3

$A = n \times n$ matrix $\Longrightarrow T: \mathbb{R}^n \to \mathbb{R}^n \qquad T(x) = Ax$

$$\left[\begin{array}{c|c} A & \begin{matrix} b_1 \\ \vdots \\ b_n \end{matrix} \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} \blacksquare & & & * \\ & \ddots & & \vdots \\ & & \blacksquare & * \end{array}\right] \iff \text{Pivot in every row}$$

<u>onto</u>

Is this system consistent for all possible b?

$$\left[\begin{array}{c|c} A & \begin{matrix} b_1 \\ \vdots \\ b_n \end{matrix} \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} \blacksquare & & & 0 \\ & \ddots & & \vdots \\ & & \blacksquare & 0 \end{array}\right] \iff \text{Pivot in every column}$$

<u>one-to-one</u>

How many solutions are there to $T(x) = 0$?
If only trivial solution — one-to-one

With a square matrix — if there's a pivot in every row, then there's a pivot in every column.

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

"Do not worry too much about your difficulty in mathematics, I can assure you that mine are still greater." -Albert Einstein

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Why are these two equivalent?

- The linear transformation $x \mapsto Ax$ is one-to-one.
- The equation $Ax = b$ has at least one solution for each b in $\mathbb{R}^n$.

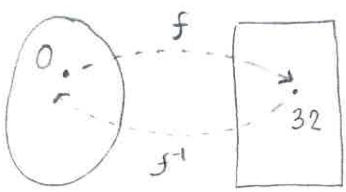
→ means the A has a pivot in each column.

* If pivot in each column & square matrix then pivot in every row too.

$$\left[\begin{array}{c|c} A & \begin{matrix} b_1 \\ \vdots \\ b_n \end{matrix} \end{array}\right] \sim \cdots \sim \left[\begin{array}{cccc|c} \bullet & & & & * \\ & \bullet & & & * \\ & & \bullet & & * \\ & & & \bullet & * \end{array}\right]$$

* Pivot in every row

* What is meant by inverse?



$$f(x) = \frac{9}{5}x + 32$$

$$f^{-1}(x) = \frac{5}{9}(x - 32)$$

find inverse

$$y = \frac{9}{5}x + 32$$
$$\Downarrow$$
$$x = \frac{9}{5}y + 32$$

$$x - 32 = \frac{9}{5}y$$
$$\Downarrow$$
$$\frac{5}{9}(x-32) = y$$
$$f^{-1}(x) = \frac{5}{9}(x - 32)$$

REMEMBER: $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

How to find the inverse of a linear transformation

Example: suppose $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\qquad T(x) = Ax$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 4x_2 \\ 5x_1 + 11x_2 \end{bmatrix}$$

* better to write using matrix multiplication

$$A = \begin{bmatrix} 2 & 4 \\ 5 & 11 \end{bmatrix} \qquad T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 4 \\ 5 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the inverse of a matrix

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 5 & 11 & 0 & 1 \end{array}\right] \xrightarrow{R_1/2} \begin{bmatrix} 1 & 2 & \frac{1}{2} & 0 \\ 5 & 11 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - 5R_1} \begin{bmatrix} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{5}{2} & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & \frac{11}{2} & -2 \\ 0 & 1 & -\frac{5}{2} & 1 \end{bmatrix}$$

$$\frac{1}{22 - 20} = \frac{1}{2}\begin{bmatrix} 11 & -4 \\ -5 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{11}{2} & -2 \\ \frac{-5}{2} & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} \frac{11}{2} & -2 \\ -\frac{5}{2} & 1 \end{bmatrix}$$

$$S: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{11}{2} & -2 \\ -\frac{5}{2} & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{11}{2}x_1 - 2x_2 \\ -\frac{5}{2}x_1 + x_2 \end{bmatrix}$$

"Beauty is the first test; there is no permanent place in the world for ugly mathematics." -Godfrey Harold Hardy

Name: ______________________ Textbook Section 2.5

$$\begin{bmatrix} 2 & -3 & 5 \\ 6 & -9 & 16 \\ -4 & 6 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 23 \\ -6 \end{bmatrix} \overset{\text{known}}{\Longrightarrow} \begin{bmatrix} 2 & -3 & 5 \\ 6 & -9 & 16 \\ -4 & 6 & -6 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 4 & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 2 & -3 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{U}$$

*lower *upper

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 4 & 1 \end{bmatrix}}_{L} \cdot \underbrace{\underbrace{\begin{bmatrix} 2 & -3 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{U}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{y} = \begin{bmatrix} 7 \\ 23 \\ -6 \end{bmatrix}$$

key: $y = Ux$ Think of $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$Ly = \begin{bmatrix} 7 \\ 23 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 4 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 23 \\ -6 \end{bmatrix} \Longrightarrow \begin{aligned} y_1 + 0y_2 + 0y_3 &= 7 \\ 3y_1 + 1y_2 + 0y_3 &= 23 \\ -2y_1 + 4y_2 + 1y_3 &= -6 \end{aligned}$$

* Solve for y_1

$y_1 = 7$
$y_2 = 2$
$y_3 = 0$

* we found y $\quad y = \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix}$

key: Let $y = Ux$

$$\left[\begin{array}{ccc|c} 2 & -3 & 5 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solve for x_1, x_2, x_3

$$\Downarrow$$

$$\begin{bmatrix} 1 & -3/2 & 5/2 & 7/2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - \frac{5}{2}R_2} \begin{bmatrix} 1 & -3/2 & 0 & -3/2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{aligned} x_1 &= -3/2 + 3/2\, x_2 \\ x_2 &= 0 + 1x_2 \\ x_3 &= 2 + 0x_2 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 0 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -6 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & -1/5 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 0 & 5 & 10 \\ 0 & 0 & -3 \end{bmatrix}}_{U}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Let $y = Ux$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & -1/5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$y_1 + 0y_2 + 0y_3 = 1 \qquad y_1 = 1$$
$$-3y_1 + y_2 + 0y_3 = 2 \qquad y_2 = 5 \qquad y = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$$
$$2y_1 - 1/5\, y_2 + 1y_3 = 3 \qquad y_3 = 2$$

$$\underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 0 & 5 & 10 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}$$

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 5 & 10 & 5 \\ 0 & 0 & -3 & 2 \end{bmatrix} \xrightarrow{R_2/5} \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix} \xrightarrow{R_1 + R_3} \begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix} \xrightarrow{R_3/-3} \begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2/3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2/3 \end{bmatrix} \xrightarrow{R_2 - 2R_3} \begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & 1 & 0 & 7/3 \\ 0 & 0 & 1 & -2/3 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 2 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 7/3 \\ 0 & 0 & 1 & -2/3 \end{bmatrix} \xrightarrow{R_1/2} \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 7/3 \\ 0 & 0 & 1 & -2/3 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 7/3 \\ -2/3 \end{bmatrix}$$

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What is an LU factorization of a matrix A?

Definition - If we can write a matrix A in the form $A = LU$, where:

- L is lower triangleular, with ones on the main diagnal,
- U is any "echelon form" (basically, in REF or RREF),

then we call this an LU factorization of A.

"A Man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator, the smaller the fraction." -Tolstoy

Introduction to Determinants

Name: ______________________________ Textbook Section 3.1

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$. In the event that $ad - bc \neq 0$, the inverse is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. We call $ad - bc$ the determinant of A, and we write $\det(A) = ad - bc$.

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible

Suppose $a \neq 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{ar_2} \begin{bmatrix} a & b \\ ac & ad \end{bmatrix} \xrightarrow{r_2 + (-c)r_1} \begin{bmatrix} a & b \\ ac - ca & ad - cb \end{bmatrix} \Longrightarrow \begin{bmatrix} a & b \\ 0 & ad - cb \end{bmatrix}$$

Is this row-reduce to ident matrix?
– Depends on # of pivots. as long as $ad \neq cb$ or $ad - bc \neq 0$

*Big Idea – A has an inverse if and only if $\det A \neq 0$

Determinant of a 3×3 matrix A

Suppose $a_{11} \neq 0$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{array}{c} \xrightarrow{a_{11} R_2} \\ \xrightarrow{a_{11} R_3} \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix}$$

$$\begin{array}{c} \xrightarrow{R_2 - a_{21} R_1} \\ \xrightarrow{R_3 - a_{31} R_1} \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} - a_{21}a_{11} & a_{11}a_{22} - a_{21}a_{12} & a_{11}a_{23} - a_{21}a_{13} \\ a_{11}a_{31} - a_{31}a_{11} & a_{11}a_{32} - a_{31}a_{12} & a_{11}a_{33} - a_{31}a_{13} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \overset{a}{a_{11}a_{12} - a_{21}a_{12}} & \overset{b}{a_{11}a_{23} - a_{21}a_{13}} \\ 0 & \overset{c}{a_{11}a_{32} - a_{31}a_{12}} & \overset{d}{a_{11}a_{33} - a_{31}a_{13}} \end{bmatrix}$$

so if $A \sim I_3 \ldots$

$$(a_{11}a_{12} - a_{21}a_{12})(a_{11}a_{33} - a_{31}a_{13}) - (a_{11}a_{32} - a_{31}a_{12})(a_{11}a_{23} - a_{21}a_{13}) \neq 0$$

CONT. NEXT PAGE

"All mathematicians share… a sense of amazement over the infinite depth and mysterious beauty and usefulness of mathematics." -Martin Gardner

3.1.6

$$\begin{bmatrix} 5 & -2 & 2 \\ 0 & 3 & -3 \\ 2 & -4 & 7 \end{bmatrix} \quad \det A =$$

$$(5)\det\begin{bmatrix} 3 & -3 \\ -4 & 7 \end{bmatrix} - (-2)\det\begin{bmatrix} 0 & -3 \\ 2 & 7 \end{bmatrix} + (2)\det\begin{bmatrix} 0 & 3 \\ 2 & -4 \end{bmatrix}$$

$$(5)\left[(3)(7)-(-4)(-3)\right] - (-2)\left[(0)(7)-(-3)(2)\right] + (2)\left[(0)(-4)-(3)(2)\right]$$

$$(5)(9) \quad - (-2)(6) \ + (2)(-6) \ = 45$$

3.1.8

$$\begin{bmatrix} 4 & 1 & 2 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{bmatrix} \quad \det A =$$

$$(4)\det\begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix} - (1)\det\begin{bmatrix} 4 & 3 \\ 3 & 5 \end{bmatrix} + (2)\det\begin{bmatrix} 4 & 0 \\ 3 & -2 \end{bmatrix}$$

$$(4)\left[(0)(5)-(-2)(3)\right] - (1)\left[(4)(5)-(3)(3)\right] + (2)\left[(4)(-2)-(3)(0)\right]$$

$$(4)(6) - (1)(11) + (2)(-8) = -3$$

3.1.19. Explore the effect of the elementary row operations on the determinant of a matrix State the row operation and its affect on the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

switched rows
The determinant switched signs

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \qquad \begin{bmatrix} c & d \\ a & b \end{bmatrix} = cd - ab$$

3.1.20.

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Determinant of a 3×3 matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

*Mess from previous simplifies to

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$$= a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Definition: For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from row 1 of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$\det \begin{bmatrix} -2 & 6 & 3 \\ 1 & 4 & -1 \\ 2 & -3 & 0 \end{bmatrix} = \underbrace{(-2)(-3) - 6(2) + 3(-11)}_{\boxed{-39}}$$

$$(-2)\det \begin{bmatrix} 4 & -1 \\ -3 & 0 \end{bmatrix} - 6 \det \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} + 3 \det \begin{bmatrix} 1 & 4 \\ 2 & -3 \end{bmatrix}$$

IMPORTANT PATTERN

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

"He who asks a question is a fool for five minutes; he who does not ask a question remains a fool forever." - Confucius

Properties of Determinants

Name: ______________________ Textbook Section 3.2

Properties of Determinats

$$\det\begin{bmatrix} ka & kb \\ c & d \end{bmatrix} = k\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= kad - kbc$$
$$= k(ad-bc)$$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad-cb \qquad \det\begin{bmatrix} c & d \\ a & b \end{bmatrix} = cd-da$$
$$= -(ad-cb)$$

$$\det\begin{bmatrix} a+a' & b+b' \\ c & d \end{bmatrix} = (a+a')d - c(b+b') = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \det\begin{bmatrix} a' & b' \\ c & d \end{bmatrix}$$
$$ad + a'd - cb - cb'$$
$$= ad - cb + a'd - cb'$$

Theorem

If B is obtained from A by swapping two rows, then $\det B = (-1)\det A$

If B is obtained from A by multiplying a row of A by the scalar k, then $\det B = k \cdot \det A$

Suppose that $R_1 \dots R_n$ and R_1' represent rows. Then

$$\det\begin{bmatrix} R_1 + R_1' \\ R_2 \\ \vdots \\ R_n \end{bmatrix} = \det\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix} + \det\begin{bmatrix} R_1' \\ R_2 \\ \vdots \\ R_n \end{bmatrix}$$

* If a square matrix A has two identical rows then $\det A = 0$

Corollary

If row i of a (square) matrix A is replaced by "(row i) + k(row j)," then this does not change the determinant of A

$$A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} \xrightarrow{R_1 + 8R_3} \begin{bmatrix} R_1 + 8R_3 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} = B \qquad \det B = \det\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} + \underbrace{8\det\begin{bmatrix} R_3 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix}}_{=0}$$

"The Universal Zulu Nation stands to acknowledge wisdom, understanding, freedom, justice, and equality, peace, unity, love, and having fun, work, overcoming the negative through the positive, science, mathematics, faith, facts, and the wonders of God, whether we call him Allah, Jehovah, Yahweh, or Jah." -Afrika Bambaataa

$$\det\begin{bmatrix}1&2&3&4\\5&6&7&8\\9&10&11&12\\1&2&3&4\end{bmatrix} = (-1)\det\begin{bmatrix}1&2&3&4\\5&6&7&8\\9&10&11&12\\1&2&3&4\end{bmatrix}$$

If a square matrix A has two identical rows then $\det A = 0$

Cramer's Rule

Name: ______________________ Textbook Section 3.3

Cramer's Rule -

Let A be an invertible $n \times n$ matrix. For any b in $\mathbb{R}^n$, the unique solution x of $Ax = b$ has entries given by

$$x_i = \frac{\det A_i(b)}{\det A}, \qquad i = 1, 2, \ldots, n$$

Cramer's Rule for solving $A\mathbf{x} = \mathbf{b}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Notice that

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}} \cdot \underset{C}{\begin{bmatrix} 1 & 0 & x_1 & 0 \\ 0 & 1 & x_2 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & x_4 & 1 \end{bmatrix}} = \underset{M}{\begin{bmatrix} a_{11} & a_{12} & b_1 & a_{14} \\ a_{21} & a_{22} & b_2 & a_{24} \\ a_{31} & a_{32} & b_3 & a_{34} \\ a_{41} & a_{42} & b_4 & a_{44} \end{bmatrix}}$$

$AC = m$

$$\det(AC) = \det(M) \implies \underbrace{\det A \cdot \det C}_{\text{multiplicative Property}} = \det M$$

$$\det C = 1 \det \begin{bmatrix} 1 & x_2 & 0 \\ 0 & x_3 & 0 \\ 0 & x_4 & 1 \end{bmatrix}$$

$$\det C = (1)(1)\det \begin{bmatrix} x_3 & 0 \\ x_4 & 1 \end{bmatrix}$$

$$\det C = (1)(1)\left(x_3(1) - x_4(0) \right)$$

$$\det C = (1)(1)(x_3)$$

$$\det C = x_3$$

$$\det A \cdot \det C = \det M$$

$$\Downarrow$$

$$x_3$$

$$(\det A)\, x_3 = (\det M)$$

$$x_3 = \frac{\det M}{\det A}$$

"The beauty of mathematics only shows itself to more patient followers." -Maryam Mirzakhani

Example of Cramer's Rule

Solve $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 1 & 6 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 1 & 6 \\ 2 & 3 & 1 \end{bmatrix}$. Note $\det A = -21$

$$* = \begin{bmatrix} 2 & 2 & 4 \\ -1 & 1 & 6 \\ 5 & 3 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} 2 & 2 & 4 \\ -1 & 1 & 6 \\ 5 & 3 & 1 \end{bmatrix} =$$

$$2 \det \begin{bmatrix} 1 & 6 \\ 3 & 1 \end{bmatrix} - 2 \det \begin{bmatrix} -1 & 6 \\ 5 & 1 \end{bmatrix} + 4 \det \begin{bmatrix} -1 & 1 \\ 5 & 3 \end{bmatrix}$$

$$(2)(1(1)-3(6)) - (2)(-1(1)-5(6)) + (4)(-1(3)-5(1))$$

$$2(-17) + 62 + (-32)$$

$$-34 + 62 - 32 = -4$$

$$x_1 = \frac{\det(*)}{\det(A)}$$

$$\det(*) = -4$$

$$\det(A) = -21$$

$$x_1 = \frac{-4}{-21} = \frac{4}{21}$$

Vector Spaces and Subspaces

Name: ______________________________ Textbook Section 4.1

DEFINITION A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V. - closure property
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$. - commutative property
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$. - Associative Property
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$. - Additive Identity Property
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$. - Additive Inverse Property
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

[1] Technically, V is a *real vector space*. All of the theory in this chapter also holds for a *complex vector space* in which the scalars are complex numbers. We will look at this briefly in Chapter 5. Until then, all scalars are assumed to be real.

Matts' informal conception of Vector Space

Very imprecise definition that goes on bumper sticker of car

A vector space is a nonempty set V of objects called vectors, along with two operations, called addition and scalar multiplication.

1) when you add two vectors, you get another vector.
2) when you multiply a scalar times a vector, you get another vector.
3) And a bunch of other predictable stuff happens too.

ex.

$$V=\mathbb{R}^2=\left\{\begin{bmatrix} a \\ b \end{bmatrix} \middle|\ a, b \text{ are in } \mathbb{R}\right\}$$

Is each set below an example of a vector space?

The set V of all functions $f(x)$ such that $\frac{d}{dx}(f(x))=3f(x)$

e.g. $f(x)=e^{3x}$

The set $M_2(\mathbb{R})$ of all 2×2 matrices with entries in $\mathbb{R}$

$$7\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}=\begin{bmatrix} 21 & 28 \\ 35 & 42 \end{bmatrix}$$

$$c \cdot u = v$$

$$v=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad w=\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$v+w=\begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

"Mathematics is a place where you can do things which you can't do in the real world." -Marcus du Sautoy

Name: ______________________ Textbook Section 4.1

Theorem. (Basic facts about vector spaces)

Let V be a vector space. For each vector v in V and each scalar c,

(i) $0\mathbf{v} = \mathbf{0}$, (ii) $c\mathbf{0} = \mathbf{0}$, and (iii) $(-1)\mathbf{v} = -\mathbf{v}$.

A lemma is a little "helper theorem"

Lemma

Let V be a vector space, and suppose that u and w are in V. If $u + w = u$, then $w = 0$

Proof.

Suppose $u + w = u$ Then,

$-u + (u + w) = -u + u$ Property # 5

$(-u + u) + w = 0$ Assoc property

$0 + w = 0$ Property

$w = 0$ Property # 4

want to show $0v = \mathbf{0}$

$0v = (0 + 0)v$

$\underset{\uparrow}{0v}_{u} = \underset{\uparrow}{0v}_{u} + \underset{\uparrow}{0v}_{w}$ Property 8

Since $0v + 0v = 0v$, our lemma implies $0v = \mathbf{0}$

"While physics and mathematics may tell us how the universe began, they are not much use in predicting human behavior because there are far too many equations to solve. I'm no better than anyone else at understanding what makes people tick, particularly women." -Stephen Hawking

Name: ____Textbook Section__________________

Subspaces of a vector space

Definition

Suppose V is a vector space. A **subspace** of V is a subset H of V that has the following properties:

1. The zero vector of V belongs to H.
2. H is closed under vector addition. This means that if $\mathbf{u}$ and $\mathbf{v}$ belong to H, then the sum $\mathbf{u} + \mathbf{v}$ is in H.
3. H is closed under scalar multiplication. This means that for each u in H and each scalar c, the vector cu is in H.

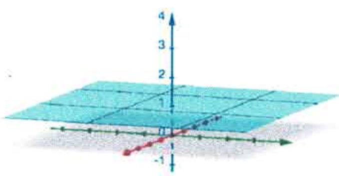
- Let $V = \mathbb{R}^3$. Is the set

$$H = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + r \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \middle| \, r, s \in \mathbb{R} \right\}$$

a subspace of V?

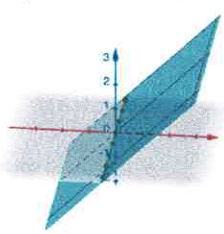
Not a subspace

Does not contain $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$



- Let $V = \mathbb{R}^3$. Is the set of all vectors of the form $r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ * H contains zero vector

(where r and s belong to $\mathbb{R}$) a subspace of V?



← H

$$V = r_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{in } H$$

$$W = r_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$V + W = r_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + r_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v + w = (r_1 + r_2) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (s_1 + s_2) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Note: This example is supposed to suggest that the span of a collection of vectors is always a subspace.

"It is generally recognized that women are better than men at languages, personal relations and multi-tasking, but less good at map-reading and spatial awareness. It is therefore not unreasonable to suppose that women might be less good at mathematics and physics." -Stephen Hawking

Name: ______________________ Textbook Section 4.2

Basic facts and definition

Theorem

Let V be a vector space. For each vector v in V and each scalar c,

(i) $0v = \mathbf{0}$, (ii) $c\mathbf{0} = \mathbf{0}$ and (iii) $(-1)v = -v$.

Section 4.2 - Null spaces, column spaces, ...

__Definition__

Suppose A in an $m \times n$ matrix. Define the null space of A, denoted by $\text{Nul } A$, by

$$\text{Nul } A = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

Null space of A is really just solution set to equation $Ax = 0$

"Function" point of view:

$A = m \times n$ $\quad T(x) = Ax \quad T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\quad$ *All vectors in $\mathbb{R}^n$ with $T(x) = 0$

Example

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 \\ -2 & -4 & 0 & 1 \end{bmatrix}$$

what is $\text{Nul } A$?

Solution to $Ax = 0$

$$\underset{2\times 4}{A} \underset{4\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & 0 & 0 \\ -2 & -4 & 0 & 1 & 0 \end{array}\right] \sim \cdots \sim \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 4 & -3/2 & 0 \\ 0 & 1 & -2 & 1/2 & 0 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

$x_1 = -4x_3 + \frac{3}{2}x_4$

$x_2 = 2x_3 - \frac{1}{2}x_4$

$x_3 = 1x_3 + 0x_4$

$x_4 = 0x_3 + 1x_4$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

"Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country." - *David Hilbert*

Name: ______________________________ Textbook Section ______________

Null Space is a Subspace

Theorem

Let A be an $m \times n$ matrix. Then Nul A is a subspace of $\mathbb{R}^n$.

Proof outline:

$Ax = 0$ and $Ay = 0 \implies A(x+y) = 0$ * Showing that Nul A is closed under vector addition

$Ax = 0$ and $c \in \mathbb{R} \implies A(cx) = 0$ * If x is in the Nul A, then cx is in Nul A

$A\mathbf{0} = 0$ * Nul space contains the zero vector

* Nul A measures how badly $T: \mathbb{R}^n \to \mathbb{R}^m$ fails to be one-to-one. If Nul A is really big, then T is very much not one-to-one.

Column Space

Definition - Suppose A is an $m \times n$ matrix. Define the column space of A, denoted by Col A, to be span of the column vectors of A.

Example. $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -3 & 4 \\ 2 & 6 & 1 \\ 3 & 5 & 0 \end{bmatrix}^{4\times 3}$ Col A = all vectors of the form $x_1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -3 \\ 6 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}$

$$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^4$$

Column Space is a Subspace

Theorem - Let A be an $m \times n$ matrix. Then Col A is a subspace of $\mathbb{R}^m$.

Proof - Col A is defined to be the span of a set of vectors in $\mathbb{R}^m$ so Col A is a subspace of $\mathbb{R}^m$.

"It's fine to work on any problem, so long as it generates interesting mathematics along the way - even if you don't solve it at the end of the day." -Andrew Wiles

Name: ______________________ Textbook Section 4.3

Basis -

$$\text{Span}\left\{\begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}6\\6\\0\end{bmatrix}\right\} \quad \text{and} \quad \text{Span}\left\{\begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$$

These two produce the same subspace $\mathbb{R}^3$ but the later doesn't contain any redundency.

Remember - Linearly Independent

Theorem - An indexed set $\{v_1, \dots v_p\}$ of two or more vectors, with $v_1 \neq 0$, is linearly dependent if and only if there exists some $j > 1$ so that v_j is a linear combination of the preceding vectors $v_1, \dots v_{j-1}$.

e.g. $v_3 = c_1 v_1 + c_2 v_2$

$$\left\{\underset{v_1}{\begin{bmatrix}3\\1\\0\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}0\\1\\0\end{bmatrix}}, \underset{v_3}{\begin{bmatrix}6\\6\\0\end{bmatrix}}, \underset{v_4}{\begin{bmatrix}1\\0\\1\end{bmatrix}}\right\} \qquad v_3 = 2v_1 + 4v_2$$

Basis - Let H be a subspace of a vector space V. (Possibly $H = V$.) An indexed set of Vectors $\mathcal{B} = \{b_1, \dots b_p\}$ is called a basis for H if

- $\mathcal{B}$ is a linearly independent set and
- the subspace spanned by $\mathcal{B}$ is H. (i.e. span $\{b_1, \dots, b_p\} = H$)

"Mathematics is the art of giving the same name to different things." -Henri Poincare

Examples Is $\mathcal{B}$ a basis?

$V = \mathbb{P}_3$, $H = V$, $\mathcal{B} = \{\overset{v_1}{1}, \overset{v_2}{x}, \overset{v_3}{x^2}, \overset{v_4}{x^3}\}$

1. Does the set $\mathcal{B}$ span all of $\mathbb{P}_3$ $\mathbb{P}$

$w = 5 - \pi x + 7.2x^2 + x^3$

$w = 5v_1 + (-\pi)v_2 + 7.2v_3 + 1v_4$

$\Rightarrow \mathcal{B}$ does span $\mathbb{P}_3$

2. Linearly Independent?

$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = \mathbf{0}$

$c_1 + c_2x + c_3x^2 + c_4x^3 = 0 + 0x + 0x^2 + 0x^3$

only possible if $c_1 = c_2 = c_3 = c_4 = 0$

So $\mathcal{B}$ is a basis

Example

$V = \mathbb{R}^3$ $H = V$, $\mathcal{B} = \{e_1, e_2, e_3\}$

Recall. $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Yes $\mathcal{B}$ is a basis.

① $V = \mathbb{P}_2$, $H = V$, $\mathcal{B} = \{3+x, 3+2x, x^2\}$

Solutions to $c_1(3+x) + c_2(3+2x) + c_3(x^2) = 0 + 0x + 0x^2$

$V_1 = (3+x)$

$V_2 = (3+2x)$

$V_3 = (x^2)$

$(3c_1 + 3c_2) + (c_1 + 2c_2)x + (c_3)x^2$

$$\begin{aligned} 3c_1 + 3c_2 &= 0 \\ c_1 + 2c_2 &= 0 \\ c_3 &= 0 \end{aligned} \Rightarrow \left\{\left[\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]\right\} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \end{aligned}$$

* Linearly Independent

Does $\mathcal{B} = \{3+x, 3+2x, x^2\}$ span $\mathbb{P}_2$?

$d_0 + d_1x + d_2x^2 = c_1(3+x) + c_2(3+2x) + c_3(x^2)$

$(3c_1 + 3c_2)1 + (c_1 + 2c_3)x + (c_3)x^2$

$$\begin{aligned} 3c_1 + 3c_2 &= d_0 \\ c_1 + 2c_3 &= d_1 \\ c_3 &= d_2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 3 & 0 & d_0 \\ 1 & 2 & 0 & d_1 \\ 0 & 0 & 1 & d_2 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

Pivot in every row

It does span $\mathbb{P}_2$

Name: ______________________ Textbook Section 4.3 cont.

Example

Is $\mathcal{B} = \{1 + 2x^2, 2 - x + x^2, 1 + x + 5x^2\}$ linearly independent?

Solutions to $c_1 v_1 + c_2 v_2 + c_3 v_3 = \mathbf{0}$

$$c_1(1+2x^2) + c_2(2-x+x^2) + c_3(1+x+5x^2) = 0 + 0x + 0x^2$$

$$(c_1 + 2c_2 + c_3) + (-c_2 + c_3)x + (2c_1 + c_2 + 5c_3)x^2 = 0 + 0x + 0x^2$$

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 0 \\ -c_2 + c_3 &= 0 \\ 2c_1 + c_2 + 5c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 1 & 5 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Not linearly independent, becaue there is a free variable

Does $\mathcal{B} = \{1 + 2x^2, 2 - x + x^2, 1 + x + 5x^2\}$ Span $\mathbb{P}_2$

$$d_0 + d_1 x + d_2 x^2 = c_1(1+2x^2) + c_2(2-x+x^2) + c_3(1+x+5x^2)$$

$$d_0 + d_1 x + d_2 x^2 = (c_1 + 2c_2 + c_3)x^0 + (0c_1 - c_2 + c_3)x + (2c_1 + c_2 + 5c_3)x^2 ?$$

$$\left\{\begin{aligned} 1c_1 + 2c_2 + 1c_3 &= d_0 \\ 0c_1 - 1c_2 + 1c_3 &= d_1 \\ 2c_1 + 1c_2 + 5c_3 &= d_2 \end{aligned}\right\} \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & d_0 \\ 0 & -1 & 1 & d_1 \\ 2 & 1 & 5 & d_2 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Name: ______________________ Textbook Section 4.3 cont.

Take a matrix A, and find the basis of Nul A and Col A.

How to find for Nul $A = \{x \mid Ax = 0\}$

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 & 2 \\ 2 & -6 & -3 & 1 & 1 \\ -3 & 8 & 2 & -3 & 0 \end{bmatrix} \quad \text{Solutions to } Ax = 0? \quad A \sim \begin{bmatrix} 1 & 0 & 6 & 5 & -4 \\ 0 & 1 & 5/2 & 3/2 & -3/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -6x_3 - 5x_4 - 4x_5 \\ -5/2\,x_3 - 3/2\,x_4 + 3/2\,x_5 \\ 1x_3 + 0x_4 + 0x_5 \\ 0x_3 + 1x_4 + 0x_5 \\ 0x_3 + 0x_4 + 1x_5 \end{bmatrix} \Rightarrow x_3 \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 3/2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

*Span

Check if span set is linearly Independent

$$\mathcal{B} = \left\{ \underset{v_1}{\begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 4 \\ 3/2 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$$

why are they linearly independent?

Suppose $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

what is the 4^{th} entry of this?

$$c_1 0 + c_2 1 + c_3 0 = 0$$

How to find basis for Col B (in a special case)

$$\text{Suppose } B = \begin{bmatrix} 1 & 0 & 3 & -4 & 0 & 5 \\ 0 & 1 & 2 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix}$$

+ alread in reduced row echelon form.

$$\text{Col } B = \text{span of}\{\text{all column vectors}\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 2 \\ 0 \end{bmatrix} \right\}$$

*notice that we can build some of the columns with others.

$$\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$b_3 = 3b_1 + 2b_2$ $\quad$ $b_6 = 5b_1 + 3b_2 + 2b_5$

$b_4 = -4b_1 + b_2$

Basis for Col B is just {Pivot columns}

Name: ______________________ Textbook Section 4.4

Important point : A vector space can have lots of different bases.

Example : Both $\mathcal{E}$ and $\mathcal{B}$ are bases for $\mathbb{R}^2$

$$\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \{e_1, e_2\} \qquad \mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} = \{v_1, v_2\}$$

$$\begin{bmatrix} 5 \\ -5 \end{bmatrix} = c_1 e_1 + c_2 e_2 ?$$
$$= 5e_1 + (-5) e_2$$

$$\begin{bmatrix} 5 \\ -5 \end{bmatrix} = c_1 v_1 + c_2 v_2$$

* This is possible because v_1 & v_2 are basis.

Theorem - (A basis allows you to write every vector in exactly one way) Let $B = \{b_1, \dots, b_n\}$ be a basis for a vector space V. Then for each x in V, there is a unique set of scalars $c_1, \dots c_n$ so that

$$x = c_1 b_1 + \dots + c_n b_n.$$

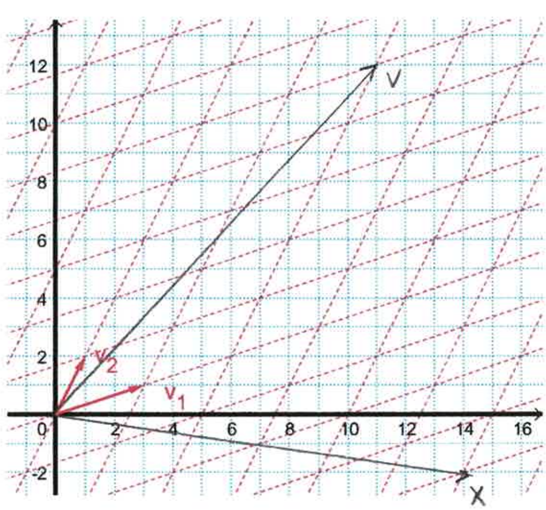
Proof. Suppose x is a vector in V. There are two things we must show.

1. Existence of scalar ?
   Possible ? Because $\{b_1, \dots b_n\}$ spans V.

   Forces $\begin{pmatrix} k_1 = 0 & k_2 = 0 & k_n = 0 \\ c_1 - d_1 = 0 & c_2 - d_2 = 0 & c_n - k_n = 0 \end{pmatrix}$

2. Uniqueness (no other possible scalars) ?

   suppose $x = c_1 v_1 + \dots c_n v_n$
   also $x = d_1 v_1 + \dots d_n v_n$
   Top-Bottom $0 = (c_1 - d_1) v_1 + \dots (c_n - d_n) v_n$
   $0 = k_1 v_1 + k_2 v_2 + k_n v_n$



$V = \mathbb{R}^2$

$$\mathcal{B} = \left\{ \begin{array}{cc} v_1 & v_2 \\ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, & \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{array} \right\}$$

$$\mathbf{x} = \begin{bmatrix} 14 \\ -2 \end{bmatrix} \qquad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 6 \\ -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} 11 \\ 12 \end{bmatrix} \qquad [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Technology is a word that describes something that doesn't work yet. – Douglas Adams

Name: ______________________________ Textbook Section ________________

Example 2

$$B = \{\underset{v_1}{2 + x^2},\ \underset{v_2}{-1 + 3x},\ \underset{v_3}{x + 2x^2}\}$$ is a basis for the vector space $\mathbb{P}_2$

$u = 1 + 11x - 5x^2$ $\qquad$ $W = 6 - 13x + 4.5x^2$

$$[u]_B = \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix} \begin{array}{l} v_1 \\ v_2 \\ v_3 \end{array} \qquad [W]_B = \begin{bmatrix} 1/2 \\ -5 \\ 2 \end{bmatrix}$$

$$1 + 11x - 5x^2 = 3v_1 + 5v_2 + (-4)v_3$$

$$6 - 13x + 4.5x^2 = \tfrac{1}{2}v_1 + (-5)v_2 + 2v_3$$
$$= \tfrac{1}{2}(2 + x^2) + (-5)(-1 + 3x) + 2(x + 2x^2)$$

Definition

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The *coordinates* of $\mathbf{x}$ relative to $\mathcal{B}$ are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

We use the notation $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. (Notice that $[\mathbf{x}]_{\mathcal{B}} \in \mathbb{R}^n$.)

Two standard questions – suppose V is a vector space with basis B

1) Given $[w]_B$, what is w? $\quad [w]_B = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$ $\quad$ e.g. $V = \mathbb{R}^2$, $B = \left\{ \underset{v_1}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \right\}$

$$w = 7v_1 + 5v_2$$
$$w = 7\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$w = \begin{bmatrix} 26 \\ 17 \end{bmatrix}$$

2) Given v in V, find $[v]_B$. $\quad v = \begin{bmatrix} 12.5 \\ 5 \end{bmatrix}$ $\quad B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$$\begin{bmatrix} 12.5 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Longrightarrow \left[\begin{array}{cc|c} 3 & 1 & 12.5 \\ 1 & 2 & 5 \end{array}\right] \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 1 & 12.5 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 1 & 12.5 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -2.5 \end{array}\right] \xrightarrow{R_2(-\frac{1}{5})} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 1/2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1/2 \end{array}\right]$$

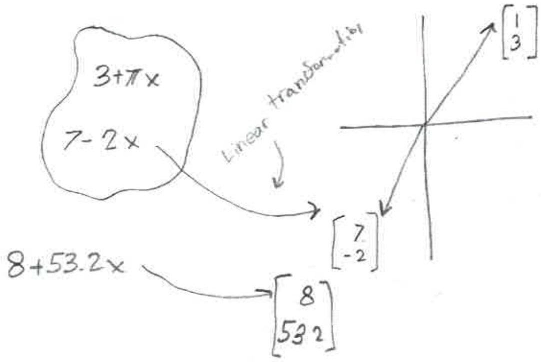
$c_1 = 4$
$c_2 = 1/2$

It has become appallingly obvious that our technology has exceeded our humanity. – Albert Einstein

Name: ______________________ Textbook Section Cont 4.4

$[\ ]_B$ gives us a function from V to $\mathbb{R}^n$

Example: $V = \mathbb{P}_1$, $B = \{1, x\}$



$$B = \{\overset{b_1}{1+t^2},\ \overset{b_2}{t+t^2},\ \overset{b_3}{1+2t+t^2}\} \text{ basis for } \mathbb{P}_2$$

Define $T: \mathbb{P}_2 \longrightarrow \mathbb{R}^3$ in the usual way (using B)

$$T(1+4t+7t^2) = c_1(1+t^2) + c_2(t+t^2) + c_3(1+2t+t^2)$$

$$= (c_1+c_3) + (c_2+2c_3)t + (c_1+c_2+c_3)t^2$$

$$\left\{\begin{array}{l} c_1 + \quad + c_3 = 1 \\ \quad c_2 + 2c_3 = 4 \\ c_1 + c_2 + c_3 = 7 \end{array}\right\}$$

Can show $1+4t+7t^2 = 2b_1 + 6b_2 - b_3$

$c_1 = 2 \quad c_2 = 6 \quad c_3 = -1$

$$T(1+4t+7t^2) = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$$

$B = \{1+t^2,\ t+t^2,\ 1+2t+t^2\}$ basis for $\mathbb{P}_2 = V$

Define $T: \mathbb{P}_2 \longrightarrow \mathbb{R}^3$ as before (using B)

T is onto. Find f with $T(f) = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

Can we find f?

$T(f) = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

means $f = 3(1+t^2) + 2(t+t^2) + (-1)(1+2t+t^2)$

$f = 2 + 0t + 4t^2$

$T(2+4t^2) = \left[2+4t^2\right]_B = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

$f = 3b_1 + 2b_2 + (-1)b_3$

Use an inverse matrix to find $[x]_B$ for the given x and B

4.4.11 - $B = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$, $x = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

$[x]_B = P_B^{-1} x$

$\begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}$ $\qquad \frac{1}{(3)(6)-(-4)(-5)} \begin{bmatrix} 6 & 4 \\ 5 & 3 \end{bmatrix}$

$\frac{1}{-2} \begin{bmatrix} 6 & 4 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -\frac{5}{2} & -\frac{3}{2} \end{bmatrix}$

$\begin{bmatrix} -3 & -2 \\ -5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{matrix} (-3)(2) + (-2)(-6) \\ (-\frac{5}{2})(2) + (-\frac{3}{2})(-6) \end{matrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

$[x]_B = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

Find the vector x determined by give coordinate Vector $[x]_B$ and the given basis B

Is $\mathbb{P}_n$ basically the same thing as $\mathbb{R}_n$?

Polynomials $\qquad$ Column vectors

4.4.1 $\quad B = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$ $\quad [x]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$X = (5)v_1 + (3)v_2$

$X = 5\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 3\begin{bmatrix} -4 \\ 6 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 15 \\ -25 \end{bmatrix} + \begin{bmatrix} -12 \\ 18 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix} \qquad X = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

Name: ______________________ Textbook Section 4.5

If V has a basis $\mathcal{B} = \{b_1 \ldots b_n\}$, then every basis for V will contain n vectors

This value n is called the dimension of V

Theorem 9: If a vector space V has a basis $\mathcal{B} = \{b_1 \ldots b_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem 10: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Example: (a) find a basis, and (b) state the dimension

4.5.1 $\begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix}$: s, t in $\mathbb{R}$ dim $H = 2$

$V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

4.5.6 $\begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix}$: a, b, c in $\mathbb{R}$

$V_1 = \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}$ $V_2 = \begin{bmatrix} 6 \\ 2 \\ 5 \\ 1 \end{bmatrix}$ $V_3 = \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix}$

dim $H = 3$

Find the dimension of the subspace spanned by the given vectors.

$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix} \underset{\text{rref}}{\sim} \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

free variable

Suppose $B = \begin{bmatrix} 1 & 0 & 3 & 2 & 0 \\ 0 & 1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ Col B dim Nul $B = 2$

Col B = Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

dim Col $B = 3$

4.5.14

$$A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

dimNul A = free variables = 3

dim Col A = pivot positions = 3

Name: ________________________________ Textbook Section 4.6

Review -

A basis for a vector space V is a set that is linearly independent and spans V.

If a linearly independent set does not span V, you can enlarge the set (increasing the span) so that the new set is still linearly independent.

If a set is not linearly independent, you can remove a vector without changing the span.

If A matrix, the set of all linear combinations of the row vectors is called the row space of A, and is denoted Row A.

Comment: $A = \begin{bmatrix} 1 & 0 & 3 & 2 & 0 \\ 0 & 1 & 4 & -1 & 0 \\ 1 & 1 & 7 & 1 & 1 \end{bmatrix} \Longrightarrow$ $r_1 = (1, 0, 3, 2, 0)$
$r_2 = (0, 1, 4, -1, 0)$
$r_3 = (1, 1, 7, 1, 1)$

Comment: Row A is a vector (subspace) of $\mathbb{R}^5$

The rank of A is the dimension of the column space of A

Row A = Span $\{r_1, r_2, r_3\}$ dim Row $A = 3$?

*make sure that no other row can be build from one another else it would be different.

We define the rank of a matrix A to be the dimension of the column space A.

$$M = \begin{bmatrix} 1 & 0 & 3 & 2 & 0 \\ 0 & 1 & 4 & -1 & 0 \\ 1 & 1 & 7 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 2 & 0 \\ 0 & 1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Rank M = dim Col $M = 3$

Col M = range T

Name: ______________________ Textbook Section 5.1

Definition of eigenvector and eigenvalue

Definition - Suppose A is an $n \times n$ matrix. A nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda \mathbf{x}$ for some scalar λ is called an eigenvector. The scalar λ is called an eigenvalue of A.

If $A\mathbf{x} = \lambda \mathbf{x}$, we say that $\mathbf{x}$ is an eigenvector corresponding to λ.

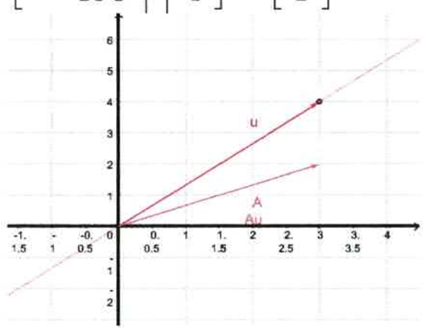
$$A = \begin{bmatrix} 7 & 1 \\ 6 & 8 \end{bmatrix} \quad X = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \lambda = 5$$

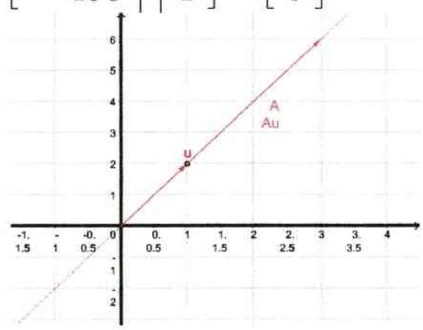
X - eigenvector

$\lambda = 5$ - eigenvalue

$$Ax = \begin{bmatrix} 7 & 1 \\ 6 & 8 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix} = 5\begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad Ax = 5x$$

$$\begin{bmatrix} -3 & 3 \\ -10 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} -3 & 3 \\ -10 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$





Finding the eigenvectors algebraically

Let $A = \begin{bmatrix} -3 & 3 \\ -10 & 8 \end{bmatrix}$. Suppose we know $\lambda = 3$ is an eigenvalue of A.

Can we find a corresponding eigenvector?

$Av = 3v$ (for some v)

$Av - 3v = 0$

$Av - (3I_2)v = 0$

$(A - 3I_2)v = 0$

$$\underbrace{(A - 3I_2)}_{B} = \begin{bmatrix} -3 & 3 \\ -10 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} -6 & 3 \\ -10 & 5 \end{bmatrix}}_{B}$$

$Bv = 0$ find v

$$\left[\begin{array}{cc|c} -6 & 3 & 0 \\ -10 & 5 & 0 \end{array}\right] \xrightarrow{R_2 - \frac{5}{3}R_1} \left[\begin{array}{cc|c} -6 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1(-\frac{1}{6})} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = \frac{1}{2}x_2$

$x_2 = x_2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

* Lots of eigenvectors! $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2.5 \\ 5 \end{bmatrix},$

"The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful" Georg Cantor

$A = \begin{bmatrix} -3 & 3 \\ -10 & 8 \end{bmatrix}$ $\lambda = 2$

Can we find a corresponding eigenvector

$Av = 2v$ find V

$Av - 2I_v = 0$

$(A - 2I_v)v = 0$

$$\begin{bmatrix} -3 & 3 \\ -10 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ -10 & 6 \end{bmatrix}$$

$B = \begin{bmatrix} -5 & 3 \\ -10 & 6 \end{bmatrix}$

$$Bv = 0 \longrightarrow \left[\begin{array}{cc|c} -5 & 3 & 0 \\ -10 & 6 & 0 \end{array}\right] \rightarrow \begin{bmatrix} -5 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -3/5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = (3/5)x_2$

$x_2 = \text{free}(x_1)$

$$V = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3/5 \\ 1 \end{bmatrix}$$

5.1.2 Is $\lambda = -2$ an eigenvalue of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$? Why or why not

$(A - -2I_2)$

$$\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

$\det \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} = 9 - 9 = 0$ yes $\lambda = -2$ is an eigenvalue

Name: ______________________________ Textbook Section ______________

Suppose A is an $n \times n$ matrix and λ is an eigenvalue for A.

- If u and v are eigenvectors corresponding to λ, then so is $u + v$

  $Au = \lambda u$
  $Av = \lambda v$
  $A(u+v) = Au + Av = \lambda u + \lambda v$
  $\lambda(u+v)$

- If v is an eigenvector corresponding to λ and $c \in \mathbb{R}$, then cv is also an eigenvector

- $A0 = 0 = \lambda 0$ (0 behaves like an eigenvector)

What does this say about the set of all v with $Av = \lambda v$?

That set forms a (vector) subspace of $\mathbb{R}^n$

Theorem (Eigenvectors for distinct eigenvalues are independent)

If $v_1, \dots v_r$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_r$, then the set $\{v_1, \dots v_r\}$ is linearly independent.

Name: ______________________________________ Textbook Section__________________

Reminder - Suppose A is an $n \times n$ matrix. A nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda \mathbf{x}$ for some Scalar λ is called an eigenvector. The scalar λ is called an eigenvalue of A.

Things we know about det

1) A is invertible if $\det A \neq 0$
2) $\det(AB) = (\det A)(\det B)$
3) $\det(A^T) = \det A$
4) If A is a triangular, then $\det A$ is the product of the entires on the main diagonal of A.
5) For an $n \times n$ matrix B, you can use det to check if there is a vector $\mathbf{x}$ in $\mathbb{R}^n$ with $B\mathbf{x} = \mathbf{0}$.

* If $\det B = 0$, then can find nonzero x with $Bx = 0$

A = matrix $(n \times n)$. λ is an eigenvalue of A if $Ax = \lambda x$ for some $x \in \mathbb{R}^n$

$$Ax = \lambda I_n x \iff Ax - \lambda I_n x = 0 \iff \underbrace{(A - \lambda I)}_{B} x = 0$$

$$Bx = 0$$

$$A - \lambda I_2 = \begin{bmatrix} 4-\lambda & -2 \\ 5 & -7-\lambda \end{bmatrix} \qquad \det(A - \lambda I_2) = \left[(4-\lambda)(-7-\lambda)\right] - \left[(5)(-2)\right]$$

* what needs to happen for $\det(A - \lambda I_2)$ to be zero (0)?

$$\left[(4-\lambda)(-7-\lambda)\right] + 10 = 0$$

$$-28 - 4\lambda + 7\lambda + \lambda^2 + 10 \xrightarrow{\text{Simlify}} \underbrace{\lambda^2 + 3\lambda - 18}_{\text{Characteristic Polynomial}} = 0$$

$$\det(A - \lambda I_2) = \lambda^2 + 3\lambda - 18$$

$$0 = (\lambda + 6)(\lambda - 3)$$

$$\lambda = -6, 3$$

$$\lambda = -6 \text{ or } \lambda = 3$$

Ex. $A = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix}$ $\lambda = 3$

$A - \lambda I_2 \Rightarrow \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 5 & -10 \end{bmatrix}$

$\det(A - \lambda I_2) = 0$

$\det \begin{bmatrix} 1 & -2 \\ 5 & -10 \end{bmatrix} \Rightarrow [(1)(-10)] - [(-2)(5)]$

$(-10) - (-10) = 0$

$\lambda = -6$

$A = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix}$ $\begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix} - \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix} = \begin{bmatrix} +10 & -2 \\ 5 & -1 \end{bmatrix}$

$\det(A - \lambda I_2) = 0$

$\det \begin{bmatrix} +10 & -2 \\ 5 & -1 \end{bmatrix} \Rightarrow [(+10)(-1)] - [(5)(-2)] =$

$-10 - -10 = 0$

5.2.8 Find the characteristic polynomial and the eigenvalues of the matrices

$\begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$

REMEMBER
$\det(A - \lambda I_2) = 0$

$\begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 7-\lambda & -2 \\ 2 & 3-\lambda \end{bmatrix}$

$\det \begin{bmatrix} 7-\lambda & -2 \\ 2 & 3-\lambda \end{bmatrix}$

REMEMBER
$\det A = ad - bc$

$(7-\lambda)(3-\lambda) - (-2)(2) = 0$

$21 - 7\lambda - 3\lambda + \lambda^2 + 4 = 0$

↓ combine like terms

$\lambda^2 - 10\lambda + 25 = 0$ * characteristic polynomial

$(\lambda - 5)(\lambda - 5) = 0$

eigenvalue $\lambda = 5$ multiplicity 2

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Theorem: If A is an $n \times n$ matrix, then a scalar λ is an eigenvalue of A if and only if λ is a solution to the characteristic equation.

$$\det(A - \lambda I) = 0.$$

Solve for λ to find eigenvalue!

Remember $\lambda I_3 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 3 & -2 \\ 1 & 5 & -2 \\ 1 & 3 & 0 \end{bmatrix} \qquad A - \lambda I_3 = \begin{bmatrix} 3-\lambda & 3 & -2 \\ 1 & 5-\lambda & -2 \\ 1 & 3 & -\lambda \end{bmatrix}$$

$$\det(A - \lambda I_3) = (3-\lambda)\det\begin{bmatrix} 5-\lambda & -2 \\ 3 & -\lambda \end{bmatrix} - 3\det\begin{bmatrix} 1 & -2 \\ 1 & -\lambda \end{bmatrix} + (-2)\det\begin{bmatrix} 1 & 5-\lambda \\ 1 & 3 \end{bmatrix}$$

$$\det(A - \lambda I_3) = -\lambda^3 + 8\lambda^2 - 20\lambda + 16 = 0$$

$\downarrow$ factorization

$$(-1)(\lambda - 4)(\lambda - 2)^2 = 0$$

$$\lambda = 4 \quad \text{or} \quad \lambda = 2$$

$\lambda = 4$ eigenspace $\Rightarrow$ 1-dimensional

$\lambda = 2$ eigenspace $\Rightarrow$ ≤ 2-dimensional

Try to solve $Bx = 0$

$$\begin{bmatrix} 3-\lambda & 3 & -2 \\ 1 & 5-\lambda & -2 \\ 1 & 3 & \lambda \end{bmatrix} \qquad \lambda = 2 \quad B = \begin{bmatrix} 1 & 3 & -2 \\ 1 & 3 & -2 \\ 1 & 3 & -2 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -3x_2 + 2x_3$$
$$x_2 = x_2 + 0x_3$$
$$x_3 = 0x_2 + x_3$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}}_{V_1} + x_3 \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{V_2}$$

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$ basis for $\lambda = 2$ eigenspace of A.

Can different

"Do not worry too much about your difficulty in mathematics, I can assure you that mine are still greater." -Albert Einstein

Name: ______________________________ Textbook Section ______________

Can different matrices have same characteristic equation?

$A = \begin{bmatrix} 2 & 7 \\ 0 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 18 & 48 \\ -5 & -13 \end{bmatrix}$

There's systematic way to make examples like this.

Use $Q = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ Notice that $B = Q^{-1} A Q$

$A - \lambda I = \begin{bmatrix} 2-\lambda & 7 \\ 0 & 3-\lambda \end{bmatrix}$ $\quad \det(*) = (2-\lambda)(3-\lambda)$

$\det(B - \lambda I) = \det(Q^{-1} A Q - \lambda I)$

$= \det(Q^{-1} A Q - Q^{-1}(\lambda I) Q)$

$= \det(Q^{-1}(A - \lambda I) Q)$

$= \det(M \; N \; P) = \det(M)\det(N)\det(P)$

$\det(B - \lambda I) = \det(Q^{-1}) \cdot \det(A - \lambda I) \det Q$

$= \det Q^{-1} \cdot \det Q \cdot \det(A - \lambda I)$

$= \det(Q^{-1} Q) \cdot \det(A - \lambda I)$

$= \det(I) \cdot \det(A - \lambda I)$

$= 1 \cdot \det(A - \lambda I)$

$* \det(B - \lambda I) = \det(A - \lambda I)$

"Beauty is the first test; there is no permanent place in the world for ugly mathematics." -Godfrey Harold Hardy

Name: ____________________ Textbook Section 5.3 Diagonalization

Diagonal Matrices

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & 9 \end{bmatrix} \qquad D^7 = \begin{bmatrix} 5^7 & 0 & 0 \\ 0 & \pi^2 & 0 \\ 0 & 0 & 9^2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 5^2 & 0 & 0 \\ 0 & \pi^2 & 0 \\ 0 & 0 & 9^2 \end{bmatrix}$$

Nearly diagonal matrices $\qquad A = PDP^{-1}$

$$A = \begin{bmatrix} -2 & 3 \\ -18 & 13 \end{bmatrix} \qquad \underset{A}{\begin{bmatrix} -2 & 3 \\ -18 & 13 \end{bmatrix}} = \underset{P}{\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}}\underset{D}{\begin{bmatrix} 4 & 0 \\ 0 & 7 \end{bmatrix}}\underset{P^{-1}}{\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}}$$

$A^8 = (PDP^{-1})(PDP^{-1})\dots(PDP^{-1})$ or $P(D^8)P^{-1}$

Definition (diagonalizable)

An $n \times n$ matrix A is diagonalizable if $A = PDP^{-1}$ for some diagonal matrix D and some invertible matrix P.

Theorem

A $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$
* diagonalizable
* diagonal

$$C = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$
* not diagonalizable

"The study of mathematics is apt to commence in disappointment." Alfred North Whitehead

$$M = \begin{bmatrix} -3 & 2 \\ -15 & 8 \end{bmatrix} = \overset{P\ D\ P^{-1}}{\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}}$$

Eigenvalues $\lambda_1 = 2 \quad \lambda_2 = 3$

$\lambda = 2$ is eig value

$$(M - 2I)x = 0$$

$$\begin{bmatrix} -5 & 2 \\ -15 & 6 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} -5 & 2 & 0 \\ -15 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2/5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 2/5\, x_2$

$x_2 = x_2$

Eigenvectors

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2/5 \\ 1 \end{bmatrix} \quad \text{Chose } x_2 = 5 \quad \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Eigenspace $\lambda = 3 = \text{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$

---

* Find P^{-1}

$$P = \begin{bmatrix} -3/4 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -3/4 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{-3/4 - 1} \begin{bmatrix} 1 & -1 \\ -1 & -3/4 \end{bmatrix} \Rightarrow -\frac{4}{7} \begin{bmatrix} 1 & -1 \\ -1 & -3/4 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -4/7 & 4/7 \\ 4/7 & 3/7 \end{bmatrix}$$

5.3.10 Diagonalize the matrices in Ex. 7-20

$$A = PDP^{-1}$$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

Find the eigenvalues

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix}$$

$$\det \begin{bmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix} = 0$$

$(2-\lambda)(1-\lambda) - 12$

$2 - 2\lambda - \lambda + \lambda^2 - 12 = 0$

$\lambda^2 - 3\lambda - 10 = 0 \longrightarrow (\lambda + 2)(\lambda - 5)$

$\lambda = -2 \qquad \lambda = 5$

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$$

Find the eigenvectors

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 4 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -3/4\, x_2$

$x_2 = x_2$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -3/4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix} \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 1 x_2$

$x_2 = x_2$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A = PDP^{-1}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} -3/4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -4/7 & 4/7 \\ 4/7 & 3/7 \end{bmatrix}$$

Name: ______________________________________ Textbook Section__________________

A process for diagonalizing a matrix (that has basis of eigenvectors)

Goal: Given $n \times n$ matrix A, find D and P so that $A = PDP^{-1}$.

$$A = \begin{bmatrix} 4 & -1 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

1) Find basis of eigenvectors
2) use basis to form columns of P
3) Use eigenvalues (in same order) as diagonal entries for D

Find basis of eigenvectors

$$B = \left\{ \underbrace{\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}}_{\lambda = 2}, \underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\lambda = 3} \right\}$$

Use basis to form colums of P

$$P = \begin{bmatrix} -1 & -1 & 1 \\ -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

use eigenvalues (in same order) as diagonal entries for D

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

"A Man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator, the smaller the fraction." -Tolstoy

Name: ______________________ Textbook Section 6.1

Definition -

For vectors $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, we define the inner product

(or dot product) of u and v by $u \cdot v = u_1 v_1 + u_2 v_2 + \dots u_n v_n = u^T v$.

Example

$v = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$w = \begin{bmatrix} -5 \\ -4 \end{bmatrix}$

$v \cdot w = 2(-5) + -3(-4) = -10 + 12 = 2$

* Dot product can tell us about length, angle, projection, ...

Theorem (Basic facts about dot product)

Let u, v, and w be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

1) $u \cdot v = v \cdot u$
2) $(u+v) \cdot w = u \cdot w + v \cdot w$
3) $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$,
4) $u \cdot u \geq 0$ and $u \cdot u = 0$ if and only if $u = 0$

Proof part 2 of theorem. $(u+v) \cdot w = u \cdot w + v \cdot w$

Write $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$. Then

$$(u+v) \cdot w = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

$$(u_1+v_1)w_1 + (u_n+v_n)(w_n) = (u_1 w_1 + u_n w_n) + (v_1 w_1 + v_n w_n)$$

Name: ____ Textbook Section____________________

Find the unit vector in the direction of the given vector

6.1-10 $\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$

$$\|v\|^2 = (-6)^2 + 4^2 + (-3)^2 = 61$$

$$\|v\| = \sqrt{61}$$

The length or (norm) of **V**. And our unit vector **u** is given by:

$$u = \frac{1}{\sqrt{61}} V = \begin{bmatrix} \frac{-6}{\sqrt{61}} \\ \frac{4}{\sqrt{61}} \\ \frac{-3}{\sqrt{61}} \end{bmatrix}$$

Given a vector like $V = \begin{bmatrix} 2 \\ \sqrt{20} \\ 5 \end{bmatrix}$, find a vector in the same direction as **V**, but with a length of 3.

$$\|v\| = \sqrt{V \cdot V} = \sqrt{2^2 + (\sqrt{20})^2 + 5^2}$$

$$\|v\| = \sqrt{4 + 20 + 25} = \sqrt{49} = 7$$

Can make a vector of length one (1) in the same direction, by taking the inverse of the length

$$\underbrace{(3)}_{\text{length wanted}} \underbrace{\frac{1}{7}}_{\text{multiplicitive inverse of length}} \underbrace{\begin{bmatrix} 2 \\ \sqrt{20} \\ 5 \end{bmatrix}}_{\text{vector } V}$$

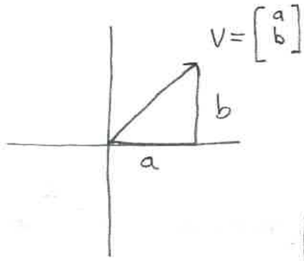
$$\frac{3}{7} \begin{bmatrix} 2 \\ \sqrt{20} \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 6/7 \\ \frac{3\sqrt{20}}{7} \\ \frac{15}{7} \end{bmatrix}$$

Technology frightens me to death. It's designed by engineers to impress other engineers. And they always come with instruction booklets that are written by engineers for other engineers – which is why almost no technology ever works. – John Cleese

Name: ______________________________ Textbook Section______________

Geometry - length

How would you compute the length of a vector in $\mathbb{R}^2$ (or even $\mathbb{R}^3$)?



Size of $v = \|v\|$

$\|v\|^2 = \sqrt{a^2+b^2}^{\,2}$

$$\boxed{\|v\|^2 = a^2+b^2 = v \cdot v}$$

* In $\mathbb{R}^4$ (or bigger), we actually treat this as a definition

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = a^2+b^2$$

Common problem: Given a vector like $v = \begin{bmatrix} 2 \\ \sqrt{20} \\ 5 \end{bmatrix}$, find a vector in the same direction as v, but with a length of 3.

$\|v\| = \sqrt{v \cdot v} = \sqrt{4+20+25} = \sqrt{49} = 7$

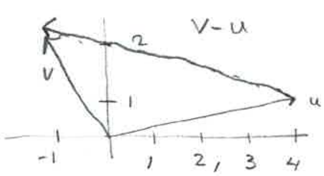
can make a vector of length 1 in the same direction (unit vector)

$u = (3)(1/7)v$ * will give you length of 3

$u = \frac{1}{7} v = \begin{bmatrix} \frac{2}{7} \\ \frac{\sqrt{20}}{7} \\ \frac{5}{7} \end{bmatrix}$ $\|u\| = 1$

Distance - Question how far apart are these two vectors.

* Final minus initial



distance $(v, u) = \|v-u\|$

$(\text{distance } (v,u))^2 = (v-u)\cdot(v-u)$

key idea (orthogonality) - Two vectors, u and v, are perpendicular (orthogonal) to each other if and only if $u \cdot v = 0$

$$\text{dist}(u,v)^2 = \|u-v\|^2 = u \cdot u - 2u \cdot v + v \cdot v$$
$$\text{dist}(u,-v)^2 = \|u+v\|^2 = u \cdot u + 2u \cdot v + v \cdot v$$

Technology... the knack of so arranging the world that we don't have to experience it. – Max Frisch

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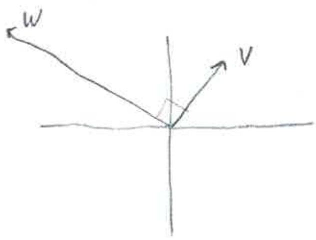


$v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$w = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$

$v \cdot w = \quad 1(-6) + 3(2) \quad = \quad -6 + 6 = 0$

Definition - For a subspace W of $\mathbb{R}^n$, define the orthogonal complement $W^\perp$ of W to be the set of all vectors in $\mathbb{R}^n$ that are orthogonal to all vectors in W.

$$W^\perp = \{x \in \mathbb{R}^n \mid x \cdot w = 0 \text{ for all } w \in W\}$$

"Do not worry too much about your difficulty in mathematics, I can assure you that mine are still greater." -Albert Einstein

Name: ______________________ Textbook Section 6.2

Definition: Orthogonal Set

A set $\{u_1, \ldots, u_n\}$ is said to be an orthogonal set if $u_i \cdot u_j = 0$ for all $i \neq j$

Example: $u_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $u_3 = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$

$u_1 \cdot u_2 = (1)(0) + (-2)(1) + (1)(2) = 0$

$u_2 \cdot u_3 = (0)(-5) + (1)(-2) + (2)(1) = 0$

$u_1 \cdot u_3 = (1)(-5) + (-2)(-2) + (1)(1) = 0$

Theorem (connection of orthogonality to linear independence)

If $S = \{u_1, \ldots, u_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and is therefore a basis for whatever subspace is spanned by S.

Theorem (Algebraic Advantage of orthogonality)

Suppose $\{u_1, \ldots, u_p\}$ is an orthogonal basis for a subspace W of $\mathbb{R}^n$. If $v \in W$, then the scalars c_i such that $v = c_1 u_1 + \ldots + c_p u_p$ are determined by

$$c_i = \frac{v \cdot u_i}{u_i \cdot u_i}$$

Ex $u_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $u_3 = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

$v = \frac{-1}{2} u_1 + \frac{7}{5} u_2 - \frac{3}{10} u_3$ How getting c_3?

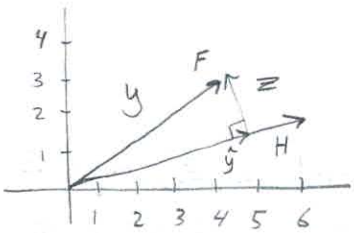
$$c_3 = \frac{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}}{\begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}} = \frac{(2)(-5) + (1)(-2) + (3)(1)}{(-5)(-5) + (-2)(-2) + (1)(1)} = \frac{-9}{30} = \frac{-3}{10}$$

Name: ____ Textbook Section ____________________

Theorem - Suppose $\{u_1 \ldots u_p\}$ is an orthogonal basis for a subspace W of $\mathbb{R}^n$. If $v \in W$, then the scalars c_i such that $v = c_1 u_1 + \ldots + c_p u_p$ are determined by $c_i =$ oops I forgot

$$v \cdot u_3 = c_3\, u_3 \cdot u_3 \implies c_3 = \frac{v \cdot u_3}{u_3 \cdot u_3}$$

Orthogonal Projection - common problem in physics



Give y & u

Goal find $\hat{y}$ and z

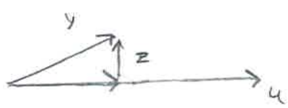
Given vectors y and u (in $\mathbb{R}^n$), can write $y = \hat{y} + z$, where $\hat{y} = cu$ and $z \cdot u = 0$

How? Force $(y - cu) \cdot u = 0$ distribute the u $y \cdot u - cu \cdot u = 0$

$$y \cdot u = cu \cdot u \qquad c = \frac{y \cdot u}{u \cdot u}$$

To project y onto u, we compute

$$\hat{y} = \left(\frac{y \cdot u}{u \cdot u}\right) u. = \text{Proj}_u(y)$$



Definition (Orthonormal set) - A set $S = \{u_1, \ldots, u_n\}$ is said to be an othonormal set if S is orthogonal and also $\|u_i\| = 1$ for all i.

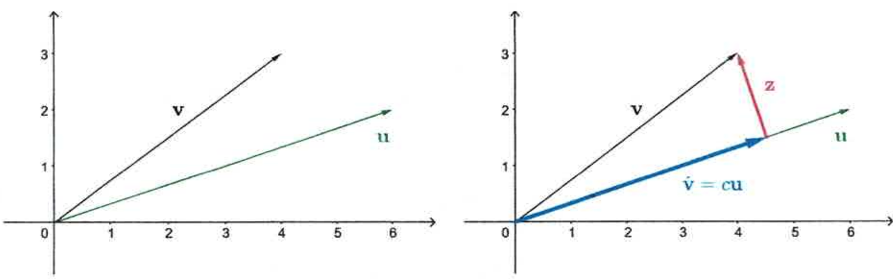
The human spirit must prevail over technology.
– Albert Einstein

Name: ______________________ Textbook Section 6.3

Reminder - The projection of a given vector v onto the line through a given vector u, is the vector $\hat{v}$ given by

$$\hat{v} = \frac{v \cdot u}{u \cdot u} u$$

We then have $v = \hat{v} + z$, where $z \cdot u = 0$. Note that $z = v - \hat{v}$

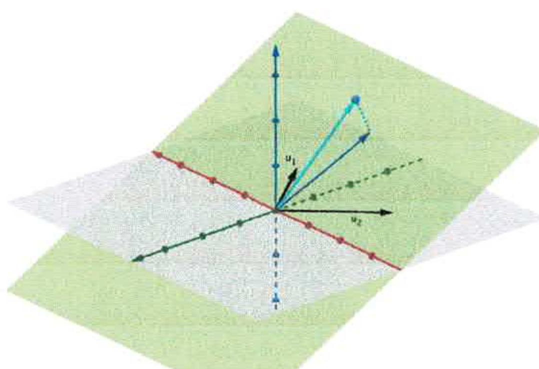


This is because: $(\mathbf{v} - c\mathbf{u}) \cdot \mathbf{u} = 0$ iff $\mathbf{v} \cdot \mathbf{u} = c\,\mathbf{u} \cdot \mathbf{u}$.

Theorem (Projecting onto a bigger subspace)

Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$. In fact, if $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is any orthogonal basis of W, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\mathbf{u}_p$$



Name for this: $\mathrm{Proj}_W(\mathbf{y})$

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 + \ldots + \frac{y \cdot u_p}{u_p \cdot u_p} u_p$$

Technological progress has merely provided us with more efficient means for going backwards. – Aldous Huxley

Name: ______________________________________ Textbook Section ____________________

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 \qquad z = y - \hat{y}$$

show that z is perp to every u_i

why is $y - \hat{y}$ is perp to u_2?

$$\left[y - \left(\frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 \right) \right] \cdot u_2 \Rightarrow y \cdot u_2 - \underbrace{\frac{y \cdot u_1}{u_1 \cdot u_1} u_1 \cdot u_2}_{=0} - \frac{y \cdot u_2}{(\cancel{u_2 \cdot u_2})} (\cancel{u_2 \cdot u_2})$$

$$= y \cdot u_2 - 0 - y \cdot u_2 \quad = 0$$

Example: Suppose $u_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

Project the vector $y = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ onto Span $\{u_1, u_2\} = W$

$u_1 \cdot u_2 = (3)(1) + (-1)(-1) + (2)(-2) = 0$ orthogonal

Calculations

$u_1 \cdot u_1 = 9 + 1 + 4 = 14$

$u_2 \cdot u_2 = 1 + 1 + 4 = 6$

$y \cdot u_1 = -3 - 2 + 6 = 1$

$y \cdot u_2 = -1 - 2 - 6 = -9$

$y = \hat{y} + z \qquad \hat{y} = c_1 u_1 + c_2 u_2$

$$c_1 = \frac{y \cdot u_1}{u_1 \cdot u_1} = \frac{1}{14}$$

$$c_2 = \frac{y \cdot u_2}{u_2 \cdot u_2} = \frac{-9}{6} = \frac{-3}{2}$$

$$\hat{y} = \frac{1}{14} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + \frac{-3}{2} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

* To get z, just compute $y - \hat{y}$

Devices are not dangerous for literature. People can be dangerous for literature. People, for example, who do not read. – László Krasznahorkai

Name: ______________________ Textbook Section 6.3 cont.

Same set-up: $\{u_1, \dots u_p\}$ orthogonal basis

Theorem - Let W be a subspace of $\mathbb{R}^n$, let y be in $\mathbb{R}^n$, and let $\hat{y} = \text{Proj}_W(y)$

Then

$$\|y - w\| > \|y - \hat{y}\|$$

for every w in W distinct from $\hat{y}$

I think that novels that leave out technology misrepresent life as badly as Victorians misrepresented life by leaving out sex. – Kurt Vonnegut

Name: ______________________________________ Textbook Section ___________________

Books may look like nothing more than words on a page, but they are actually an infinitely complex imaginotransference technology that translates odd, inky squiggles into pictures inside your head. – Jasper Fforde

The Gram-Schmidt Process

Name: ______________________ Textbook Section 6.4

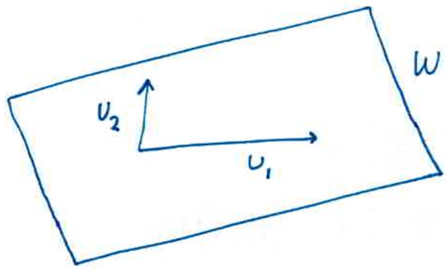
Main goal of the day

Suppose we're given a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ for a subspace W of $\mathbb{R}^n$. We wish to find a new basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ for W so that:

1. $\text{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} = \text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for every $k \geq 1$, and
2. the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are (pairwise) orthogonal.

Simple Plan

*change basis to an orthogonal basis



Let $u_1 = x_1$

$u_2 = x_2 - b u_1$

$u_3 = x_3 - c u_1 - d u_2$

$u_4 = x_4 - e u_1 - f u_2 - g u_3$

$\vdots$

Plan: $u_1 = x_1$

$u_2 = x_2 - b u_1$ How to choose b? Want $u_2 \cdot u_1 = 0$

$0 = u_2 \cdot u_1 = (x_2 - b u_1) \cdot u_1 \overset{\text{distribute}}{\Rightarrow} x_2 \cdot u_1 - b u_1 \cdot u_1$ $\qquad b = \dfrac{x_2 \cdot u_1}{u_1 \cdot u_1}$

$$u_2 = x_2 - \left(\frac{x_2 \cdot u_1}{u_1 \cdot u_1}\right) u_1$$

$u_3 = x_3 - c u_1 - d u_2$ How to choose c and d?
want $u_3 \cdot u_1 = 0$ and $u_3 \cdot u_2 = 0$

$0 = u_3 \cdot u_2$

$0 = (x_3 - c u_1 - d u_2) \cdot u_2 \Rightarrow x_3 \cdot u_2 - d u_2 \cdot u_2$

$$\boxed{d = \frac{x_3 \cdot u_2}{u_2 \cdot u_2}}$$

$$u_3 = x_3 - \left(\frac{x_3 \cdot u_1}{u_1 \cdot u_1}\right) u_1 - \left(\frac{x_3 \cdot u_2}{u_2 \cdot u_2}\right) u_2$$

We are stuck with technology when what we really want is just stuff that works. – Douglas Adams

Name: ____ Textbook Section ____________________

Gram-Schmidt Process

Suppose we're given a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ for a subspace W of $\mathbb{R}^n$. We wish to find a new basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ for W so that:

1. $\text{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} = \text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for every $k \geq 1$, and
2. the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are (pairwise) orthogonal.

Here's how:

1. Let $\mathbf{u}_1 = \mathbf{x}_1$,
2. $\mathbf{u}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right) \mathbf{u}_1$,
3. $\mathbf{u}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right) \mathbf{u}_1 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right) \mathbf{u}_2$,
4. $\mathbf{u}_4 = \mathbf{x}_4 - \left(\frac{\mathbf{x}_4 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right) \mathbf{u}_1 - \left(\frac{\mathbf{x}_4 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right) \mathbf{u}_2 - \left(\frac{\mathbf{x}_4 \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\right) \mathbf{u}_3$,
5. $\vdots$

Name: ____Textbook Section____________________

Technology is a word that describes something that doesn't work yet. – Douglas Adams

Name: ______________________________________Textbook Section___________________

It has become appallingly obvious that our technology has exceeded our humanity. – Albert Einstein

Name: ______________________________________Textbook Section__________________

Technology is just a tool. In terms of getting the kids working together and motivating them, the teacher is the most important. – Bill Gates

Name: ____________________________________Textbook Section__________________

One machine can do the work of fifty ordinary men.
No machine can do the work of one extraordinary man. – Elbert Hubbard

Name: ______________________________ Textbook Section ________________

"The beauty of mathematics only shows itself to more patient followers." - Maryam Mirzakhani

Name: ____Textbook Section____________________

Our technology forces us to live mythically. – Marshall McLuhan